Special Session 13: Shapes and Free Boundaries

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The goal of this session is to present recent results on the mathematical analysis of optimal shapes and the very connected field of free boundary problems. Regularity questions for optimal shapes/free boundaries will be discussed together with existence results, qualitative properties, numerical computations and applications to various fields.

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On a singular free boundary problem from image processing

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This talk concerns a degenerate nonlinear parabolic equation with moving boundaries which describes the technique of contour enhancement in image processing. Such problem arises from the model by Malladi and Sethian after an asympotic expantion suggested by Barenblatt: in order to recover the phenomenon of mass concentration, a singular data is imposed at the free boundary.

I will relate a joint work with Juan Luis Vázquez, where we directly analysed the singular problem and obtained (i) the well posedness for general initial datum, and (ii) the convergence of combustion type problems to the singular one.

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Using the shape Hessian to recover the geometry of an inclusion

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We consider the inverse problem of recovering the shape of an unknown inclusion inside an domain from electrostatic measurements made on the outer boundary. The fundamental questions of existence and stability of this problem are non well understood. From a numerical point of view it is known to be severely ill-posed.

Our aim is to consider this problem from a shape optimization point of view. We translate the original inverse problem as the minimization of Least Squares boundary measurements fitting. We recover the problem ill-posedness within the properties of the Hessian at minimizers of criterion. We show that no coercivity holds even for weaker norms than the differentiability norm and that the Hessian at critical shapes is compact. On some examples, we exhibit the precise spectral behavior of the Hessian operator.

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Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group H^1

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One of the most celebrated problems in geometry and calculus of variations is the Bernstein problem, which asserts that a C^2 minimal graph in R^3 must necessarily be an affine plane. In this talk we will discuss, in the simplest model of a sub-Riemannian space, the three-dimensional Heisenberg group H^1, the structure of C^2 minimal graphs with empty characteristic locus and which, on every compact set, minimize the horizontal perimeter. As a corollary of our results we obtain a positive answer to a sub-Riemannian analogue of the Bernstein problem.

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On the existence of a complete non-planar free boundary graph

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In this talk we will describe the regularity theory for certain energy minimizing free boundaries. Precisely, we consider minimizers to the energy functional $\int(|\nabla u|^2+\chi_{\{u>0\}})dx$. We will exhibit the first example of a singular energy minimizing free boundary, which occurs in dimension 7. This is the analogue of the 8-dimensional Simons cone in the theory of minimal surfaces. Its existence was proved in a joint work with D. Jerison. We will discuss a related question, that is the existence of a complete non-planar free boundary graph in dimension 8. In the theory of minimal surfaces, an example of a complete non-planar minimal graph was provided by Bombieri, De Giorgi and Giusti.

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Uniqueness and numerical analysis for Hamilton-Jacobi equations with discontinuities

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We consider the Hamilton–Jacobi equation of eikonal type

$$H(\nabla u) = f(x), \quad x \in \Omega,$$

in a bounded domain Ω with Dirichlet boundary conditions. Here, H is convex and f is allowed to be discontinuous. Under suitable assumptions on f we obtain a comparison principle for sub- and supersolutions which satisfy the above equation in a generalized viscosity sense introduced by Ishii. In order to approximate the solution, a class of finite difference schemes is considered, which are monotone, consistent and satisfy a suitable stability condition. We discuss the solvability of the discrete problem and develop an error analysis.

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What is the optimal shape of an axon?

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An *axon* is an extension from the neuron cell body that takes information away from the cell body. The role of an axons is to carry nerve impulses away from the soma to the presynaptic terminals where the impulses are transmitted to other neurons or to muscles in the case of motor neurons.

The propagation of an electric impulsion in an axon fiber follows an equation established by W. Rall in the sixties. We look for the shape of an axon which minimizes **the attenuation in time** of the signal. A good criterion should be the first eigenvalue of a spectral problem related to Rall's model.

We study this optimization problem from a mathematical and numerical point of view. By comparison with the real shape of an axon, we can discuss to know whether the criterion was a good one or not.

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Some regularity results in a shape optimization problem with perimeter

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We consider optimal shapes of the functional

$$\mathcal{E}_\lambda(\Omega) = J(\Omega) + P(\Omega) + \lambda||\Omega| - m|$$

among all the measurable subsets Ω of a given open bounded domain $D \subset \mathbf{R}^d$ where $J(\Omega)$ is some Dirichlet energy associated to Ω, $P(\Omega)$ and $|\Omega|$ being respectively the perimeter and the Lebesgue measure of Ω. We prove here that for some optimal shape, the state function associated to the Dirichlet energy is locally Lipschitz-continuous. Then we deduce the same regularity properties for the boundary of the optimal shape as in the pure isoperimetric problem (case $J \equiv 0$). We also consider the minimization of $\mathcal{E}_0$ with Lebesgue measure constraint $|\Omega| = m$.

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Geometric viscosity solutions and minimizing movements for Bernoulli's problem

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We are interested in Bernoulli's problem which can be stated as follows: given a smooth compact subset $S \subset \mathbb{R}^N$, we look for a compact subset $\Omega \supset\supset S$ which minimizes the capacity with a volume constraint. The gradient flow associated with this problem leads to the study of a nonlocal front propagation problem with a prescribed normal velocity of type $V = -1 + \mathrm{Hele-Shaw}$ term. We produce long-time geometric viscosity solutions to this evolution and prove they converge to a free boundary problem which is formally equivalent to Bernoulli's problem. Then, we study more precisely the link between these two problems introducing some minimizing movements. It allows us to prove that the energy of the flow is nonincreasing.

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Constant width bodies in dimension 3

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A body (that is, a compact connected subset K of $\mathbb{R}^n$) is said to be of *constant width* α if its projection on any straight line is a segment of length $\alpha \in \mathbb{R}_+$, the same value for all lines. We present in this talk a complete analytic parametrization of constant width bodies in dimension 3 based on the median surface: more precisely, we define a bijection between some space of functions and constant width bodies. We compute simple geometrical quantities like the volume and the surface area in terms of those functions. As a corollary we give a new algebraic proof of Blaschke's formula. Finally, we present some numerical computations base on the preceding parametrization.

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Rearrangement inequalities and applications to isoperimetric problems for eigenvalues

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Let Ω be a C^2 bounded domain of $\mathbf{R}^n$, and consider $L = -div(A\nabla) + v \cdot \nabla + V$ a second order elliptic differential operator with Dirichlet boundary condition on Ω. Define Ω^* as the Euclidean ball centered at 0 with same Lebesgue measure as Ω. We associate to L a second order elliptic operator $\widetilde{L}$ with Dirichlet boundary condition on Ω^*, obtained by a new type of rearrangement, and compare the principal eigenvalues of L and $\widetilde{L}$.

As an application we solve optimization problems for the principal eigenvalue of L when Ω, A, v and V vary under some constraints. For instance, we can impose L^p constraints on the principal eigenvalue of A (or constraints on the determinant and the trace of A) and L^p constraints on v, and a constraint on the distribution function of V.

As a particular case, we generalize the classical Rayleigh-Faber-Krahn inequality for the principal eigenvalue of the Laplace operator to the case of $-\Delta + v \cdot \nabla$ where $v \in L^\infty(\Omega)$ satisfies $\|v\|_\infty \leq \tau$ for some fixed $\tau \geq 0$.

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