Special Session 15: Multiscale analysis in Mathematical Physics

Vieri Mastropietro, Universitá di Roma "Tor Vergata", Italy

Multiscale analysis is a powerful and quite general method in mathematical physics, with application in a wide range of fields like Functional infinite dimensional integrals in Quantum Field Theory and Statistical physics, KAM theory, Spectral theory of Schroedinger operator, PDE analysis Quantum Field Theory and many others. In this session will be presented several new remarkable results based on multiscale analysis implemented in a Renormalization Group framework.

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Quasi-periodic attractors, divergent series and Borel-summability in forced dynamical systems with strong damping

Guido Gentile
Universita' di Roma Tre, Italy
gentile@mat.uniroma3.it

We consider dissipative systems with a periodic or quasi-periodic forcing, hence with possible small divisor problems, and we study existence and property of attractors with renormalization group techniques inherited from quantum field theory. The perturbation series are likely to diverge, but they can be given a meaning by introducing a suitable summation criterion which can be interpreted as a summation of self-energy graphs in quantum field theory. The re-summed series are no more analytic. In certain cases one can prove that the perturbation series are Borel-summable.

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Borel summability and Lindstedt series

Alessandro Giuliani
Princeton University, USA
giuliani@Princeton.EDU
O. Costin, G. Gallavotti and G. Gentile

We consider a quasi integrable analytic Hamiltonian system on T^d, $d \geq 3$. We review the construction of low dimensional hyperbolic KAM tori, based on iterative summation rules of the formal Lindstedt series and defining C^∞ conjugation functions of the perturbation strength. We prove Borel summability of their sums in the case of two-dimensional rotation vectors of constant type.

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A Functional Integral Representation for Many Boson Systems

Horst H. Knörrer
ETH Zürich, Switzerland
knoerrer@math.ethz.ch
T. Balaban, J. Feldman and E. Trubowitz

We are developing a set of tools and techniques for analyzing the large distance/infrared behaviour of many boson systems as the temperature tends to zero. The first tool is a functional integral representation for the grand canonical partition function and correlation functions of many bosons moving in a space X with a finite number of points. Informally, the statement of one of our main results is

$$\mathrm{Tr} e^{-\beta(H-\mu N)} = \lim_{p\to\infty} \int \prod_{\tau=\frac{\beta}{p},\frac{2\beta}{p},\cdots,\beta} d\mu_{R(p)}(\phi_\tau^*, \phi_\tau)\ e^{\mathcal{F}(\frac{\beta}{p},\phi^*,\phi)}$$

Here, H is the Hamiltonian and N is the number operator of our system of identical bosons, μ is the chemical potential and the temperature is $\frac{1}{k\beta} > 0$. For each point (x,τ) in the discrete space–time we introduce the complex variable $\phi_\tau(x)$. The measure is

$$d\mu_{R(p)}(\phi^*, \phi) = \prod_{x\in X} \left[\frac{d\phi^*(x) \wedge d\phi(x)}{2\pi\imath}\, \chi_r(|\phi(x)|)\right]$$

where, χ_r is the characteristic function of the closed interval $[0,r]$. The sequence $R(p) > 0$ that describes the large field cutoff tends to infinity at an appropriate rate as $p \to \infty$. The "action" $\mathcal{F}(\frac{\beta}{p},\phi^*,\phi)$ is similar to the one in standard physics textbooks.

The next step towards controlling the thermodynamic limit of a many boson system, is to express the temporal, ultraviolet limit $p \to \infty$ in a form suitable for an infrared renormalization group analysis. Principally, to extract an effective potential that exhibits the mechanism for symmetry breaking. This is quite subtle because the exponential of the action is highly oscillatory.

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Renormalization on Riemannian manifolds

Christoph Kopper
Ecole Politechnique, France
kopper@cpht.polytechnique.fr
Volkhard F. Mueller

We study the theory of the self-interacting scalar field on geodesically complete Riemannian manifolds of globally bounded curvature. With the aid of the Wilson-Wegner-Polchinski Flow equations of the renormalization group, we prove renormalizability of the theory to all orders in the loop expansion. We also establish bounds on the long distance decay of the perturbative Schwinger functions, as a consequence of the decay properties of the heat kernel on the manifolds considered, which were established in by Davies, Li, Cheng and Yau and others.

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Two perturbative proofs of the analyticity of the dipole gas model

Jacques Magnen
CPHT Ecole polytechnique, France
magnen@cpht.polytechnique.fr
Arif Mardin

To prove the analyticity in the coupling constant we bound each term of the perturbative expansion of the two-point correlation function of the dipole gas : i.e. the sum of all the terms containing exactly n interaction vertices.

The first proof, which is very simple, doesn't allow to estimate the behavior of the two-point function. The latter is possible with the second proof, which contains a scale analysis and a kind of large field argument ; it is thus more involved and more powerfull.

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Renormalization Group approach to PDE

Vieri Mastropietro
Universitá di Roma "Tor Vergata", Italy
mastropi@mat.uniroma2.it
G. Gentile and M. Procesi

Renormalization Group methods can be applied to a variety of PDE problems. As an example, I will discuss the problem of the constructive determination of oscillatory solutions of nonliner wave equations either in the massless and massive case. Such solutions are written as Lindstedt series and using constructive methods inspired by Quantum Field Theory and Renormalization Group their converge is proved.

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Infrared-finite algorithms in QED

Alessandro Pizzo
ETH-Zuerich, Switzerland
pizzo@phys.ethz.ch

We consider a nonrelativistic electron moving in the Coulomb field of a single nucleus of unit charge and interacting with the soft modes of the quantized electromagnetic field. Our main concern is how to rigorously control the higher order radiative corrections to the scattering amplitudes in the low energy regime (Rayleigh scattering). In fact Taylor formula is ill-defined when no infrared regularization is adopted. We develop a proper perturbation theory and we provide an asymptotic expansion up to any order in the coupling constant for the scattering amplitudes, which represents a first important step towards a rigorous analysis of metastable states. At this stage (scattering amplitudes), the main mathematical issue is the asymptotic expansion of the groundstate vector of the system. Concerning this expansion, we use a scaling analysis technique based on the iteration of the analytic perturbation.

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