Special Session 17: Reaction-Diffusion Systems and the Dynamics of Patterns

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We focus on qualitative studies of reaction-diffusion systems arising in various fields of applications such as biology, chemistry and fluid flow through porous media. Our goal is to bring together new methods and ideas for better understanding of the mathematical mechanism behind a large variety of pattern formations. Our topics include: singular limits, large time behavior, transient phenomena, blow-up in finite time, travelling waves, self-similar profiles and so on.

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On the vanishing viscosity convergence of travelling-front speeds for reaction-diffusion equations with non-convex flux

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We compare speeds of travelling-front solutions of the scalar balance law $u_t + f(u)_x = g(u)$ and the related parabolic equation $u_t + f(u)_x = \varepsilon u_{xx} + g(u)$. The (monostable) source term g has simple zeroes at 0 and 1 and is negative in between, and the flux f is (possibly) non-convex, which complicates study of the hyperbolic equation. In both cases, monotone travelling fronts connecting 0 to 1 exist for all velocities greater than or equal to some $\varepsilon-$dependent minimal value, the fronts being smooth for the parabolic and entropy solutions for the hyperbolic equations respectively. We show that as ε tends to 0, the minimal parabolic front velocity converges to the minimal hyperbolic front velocity. The proof uses comparison theorems and a variational characterisation of the minimal parabolic front velocity. As a byproduct, we find that for every $\varepsilon > 0$, the minimal parabolic front speed is greater than the minimal hyperbolic front velocity, and also derive a source-independent sufficient condition for the minimal velocity of the parabolic problem for small positive ε to be strictly greater than the value predicted by the problem linearized about the unstable equilibrium 1.

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Allen-Cahn and Cahn-Hilliard models for stress and electromigration induced surface diffusion with applications to epitaxial growth and void evolution

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We present a novel asymptotic expansion technique for second and fourth order variational inequalities and derive sharp interface models which have been first introduced by Cahn and Taylor. In addition it is demonstrated how conditions at a triple junction in a sharp interface description can be derived from a Cahn-Hilliard variational inequality.

Finally, we present a fully practical finite element approximation for Allen-Cahn and Cahn-Hilliard models taking into account electromigration, elastic effects and grain boundary motion. Applications to quantum dot formation in heteroepitaxial strained films and intergranular void evolution in microelectronic devices are discussed.

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Energy estimates for electro-reaction-diffusion systems with partly fast kinetics

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We start from a basic model for the transport of charged species in heterostructures containing the mechanisms diffusion, drift and reactions. Considering limit cases of partly fast kinetics we derive reduced models. This reduction can be interpreted as some kind of projection scheme for the weak formulation of the basic electro–reaction–diffusion system. We derive assertions concerning invariants and steady states and prove the monotone and exponential decay of the free energy along solutions to the reduced problem and to its fully implicit discrete-time version by means of the results for the basic problem. Moreover we make a comparison of prolongated quantities with the solutions of the basic model.

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Boundary layer similarity flow driven by power–law

shear: The integral equation method

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The boundary–layer similarity flow driven over a semi–infinite flat plate by a power law shear with asymptotic velocity profile $u(x,y)(y) = \beta/y(y \to \infty, \beta > 0)$ is considered. Theoretical analysis is reported to derive arange of amplitudes β for which similarity solutions exist

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2d compactness of the Néel wall

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We study the asymptotics of a family of energy-functionals coming through dimensional reduction of a three dimensional model in a thin film. We prove compactness for families of magnetizations in the energy regime corresponding to a finite number of Néel walls. The accumulation points are unit-valued divergence-free vector fields. In the case of zero-energy states, we show locally Lipschitz continuity and these states classically satisfy the principle of characteristics. Then we are interested in transition layers which connect two opposite magnetizations in a strip. We prove the optimality of the straight walls in the regime of the specific line energy of the Néel wall. In the general regime of a finite number of Néel walls, we show that $1d$ magnetizations do concentrate on a vertical lines crossing the strip. This is a joint work with Felix Otto.

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On the Daniel's and Elias' solutions of the Morisita-Shigesada et al. system

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The system

$$u_t = d_u(uu_x)_x + u(a_u - b_u u - c_u v)$$
$$v_t = d_v(vv_x)_x + v(a_v - b_v u - c_v v)$$

is considered. Travelling wave (Daniel) and compactly supported (Elias) solutions are presented. Comparison is made with corresponding non-degenerate system and with 1-equation model. A conjecture is made for the minimal speed.

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Front propagation in the Fisher equation with degenerate diffusion

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D. Hilhorst, R. Kersner and M. Mimura

Degenerate diffusion appears in models for biological invasion to take into account population density pressure. We consider the singular limit of the Fisher equation with degenerate diffusion. We establish that for compactly supported initial data, the speed of the limit interface is the minimal speed of associated travelling waves. We also exhibit exact travelling front solutions to a 2-species competition system with degenerate diffusion.

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Travelling waves of a mean curvature flow equation in periodic media

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We study a mean curvature flow equation in heterogeneous media with periodic vertical striations: $V = a(x)\kappa + b(x)$, where for a plane curve, V denotes its normal velocity, κ denotes its curvature, a and b are positive, ε-periodic functions.

Given an inclination angle $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we first prove the existence of a planar-like curve, which travels in y-direction with a constant speed c_α, and its graph is periodic oscillation of a straight line with inclination angle α.

Next, we give the estimate of travelling speed c_α for homogenization problem (as ε tends to zero). Roughly speaking $c_\alpha \approx \frac{D}{A\cos\alpha}$, where D and A are the means of $\frac{b}{a}$ and $\frac{1}{a}$, respectively.

Finally, we seek for "V"-shape travelling curves, which also travels in y-direction with speed c_α, whose graph is like letter "V", and at infinity its graph approaches the planar-like travelling curves (with speed c_α) exponentially.

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Traveling waves through a Penrose pattern

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We study front propagation for an Allen-Cahn type equation in spatially heterogeneous media having the same kind of recurrent properties as the Penrose tiling. The equation is written, typically, in the form

$$u_t = \nabla \cdot (d(x)\nabla u) + h(x)f(u).$$

As is well-known, the Penrose tiling has a quasi-periodic nature. However, contrary to the general perception, this does not mean that the coefficients $d(x), h(x)$ are quasi-periodic functions. In fact, they do not even belong to the class of almost periodic functions. Consequently, it is not immediately clear whether or not traveling waves in such media have a well-defined average speed. For a traveling wave to have the average speed, one has to check if a certain function g that depends on d and h in a certain nonlocal way has the uniform averaging property. We introduce a class of functions that is wider than that of almost periodic functions and prove that the average speed exists under such spatial heterogeneity.

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Spatially segregating patterns arising in competition-diffusion systems

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Understanding of spatial and/or temporal behaviors of ecologically interacting species is a central problem in population ecology. As for competitive interaction of ecological species, problems of coexistence or exclusion have been theoretically investigated by using different types of mathematical models. Especially, variety types of RD equations have been proposed to study spatial segregation of competing species. Recently, in order to understand the evolutional behavior of spatially segregating regions of competing species, the analytical methods called singular limits have been successfully developed. These enable us to derive evolutional equations describing internal boundaries of spatially segregating regions of competing species. In this lecture, I would like to focus on qualitative behavior of spatially segregating solutions to the normal diffusion as well as cross-diffusion-competition systems, by using singular limit analysis.

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Bifurcation structure of a 1-D Ginzburg-Landau model

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We consider the following one-dimensional Ginzburg-Landau equation with periodic boundary conditions:

$$\psi_t = D_h^2\psi + \lambda(1-|\psi|^2)\psi, \quad \psi(x+2\pi) = \psi(x),$$
$$D_h := \partial/\partial x - ih(x)$$

where ψ is a complex-valued order parameter, λ is a non-negative parameter and $h(x)$ is an arbitrarily given periodic function. This equation is a gradient equation of the energy functional with the constant $a(x)$

$$E(\psi) := \int_0^{2\pi} \left\{ \frac{1}{2}|D_h\psi|^2 + \frac{\lambda}{4}(1-|\psi|^2)^2 \right\} a(x)dx.$$

We reveal a global bifurcation structure of equilibrium states of the equation in the parameter space of (μ, λ), $\mu := (1/2\pi)\int_0^{2\pi} h(x)dx$. Namely we solve every equilibrium solution and show how it appears or terminates through a bifurcation. We also consider the gradient equation of $E(\psi)$ with nonconstant $a(x)$ and apply the center manifold theorem to the study of the local bifurcation structure near some critical point for $a(x)$ with a small variation.

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Speed of front propagation for a competition-diffusion system with variable coefficients

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We consider a competition-diffusion system with spatially periodic coefficients. We estimate the propagation speed of traveling front solutions by comparison technique and study the influence of the spatial inhomogeneity.

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Radial and nonradial steady-states with clustering layers in Allen-Cahn equations

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We consider steady-states of Allen-Cahn equation with a small diffusion coefficient ε on a ball. If the diffusion coefficient ε is very small, there appear radial steady-states with many clustering layers. Morse indices of such steady-states go to infinity if ε goes to zero. We will give the proof of this phenomena and estimate the growth rate of Morse indices. We also think about existence of nonradial solutions with many clustering layers.

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Spreading speeds of a cooperative system

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In this talk we consider a two-component cooperative system:

$$\begin{aligned} u_t &= u_{xx} + u(1 - pu + (p-1)v) \\ v_t &= v_{xx} + rv(1 + (q-1)u - qv). \end{aligned}$$

where $p > 1, q > 1$ and $r > 0$. How does the solution starting with initial conditions $H(-x)$ behave as $t \to \infty$? Here $H(x)$ is the Heaviside function. We will show that one of the components has a constant spreading speed c_1^* and the other component has the same spreading speed when its value is greater than some constant $\gamma \in (0,1)$ but spreads at a faster speed c_2^* when its value is below γ for some r. This means that one of its components develops into a stacked front as $t \to \infty$.

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Applications of the Cahn-Hilliard equation

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The Cahn-Hilliard equation was originally derived by Cahn and Hilliard in 1958 as a phenomenological model for phase separation in binary alloys. Since then, the applicability of the Cahn-Hilliard equation has become more and more apparent in numerous seemingly unrelated contexts, from the structure of biofilms to the rings of Saturn, from image processing to ecological models and thin films. We review some of these different models, explaining how the Cahn-Hilliard equation arises and what the Cahn-Hilliard results imply.

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Travelling waves for a reaction-diffusion equation with periodic nonlinearity

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We consider a reaction-diffusion equation $u_t = u_{xx} + f(u)$ on $\mathbb{R}$, which is related to a mathematical model of crystal growth. Here $f(u)$ is assumed to be a periodic function. We prove the existence of travelling waves with unbounded profiles and study the relation between the velocity and the shape of travelling waves.

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Twisted Rods and Biological Membranes: Two Nonlocal Obstacle Problems

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Two problems from vastly different application areas lead to mathematical problems that share a common feature: they both can be formulated as minimization of a relatively simple functional under a nonlocal constraint of obstacle type (a convolution complementarity problem).

When twisting an elastic rod around a cylinder, a self-contact problem arises when the rod tries to penetrate itself; after a suitable transformation this problem can be written as minimization of a functional of H^1-type over a class of functions that satisfies a convolution inequality. Combining use of the Euler-Lagrange equation with classic variational techniques allows us to determine the structure of the solution, and most importantly of the contact set. An interesting corolloary is a full characterization of the contact forces.

A simple model of planar lipid bilayers also leads to minimization over a similar class of functions, defined by a convolution inequality. In this case the functional to be minimized contains no derivatives, and consequently a main difficulty in the analysis is the lack of inherent regularity of stationary points. By an interesting application of the Sobolev-norm characterization by Bourgain, Brezis, and Mironescu we prove nonetheless that stationary points themselves have H^1 regularity.

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Analysis of a pore scale model for dissolution and precipitation in porous media

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We discuss a pore scale model for precipitation and dissolution in porous media, where dissolved cations and anions are transported by the fluid flowing through the pores of the medium. These ions can precipitate and form a crystalline solid, which is attached to the surface of the porous skeleton, and thus is immobile. The reverse reaction of dissolution is also possible.

A special attention is paid to the dissolution rate, which is modeled as a multivalued graph. We give some qualitative properties of the model for general domains, and then consider simpler geometries. In particular, if the void space is a strip, with dissolution and precipitation occurring at the lateral boundaries, we investigate the formation of a dissolution front.

Further, as the ratio between the thickness and the length of the strip vanish, we end up with the upscaled transport–reaction model proposed by C.J. van Duijn and P. Knabner.

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Mathematical analysis of a model describing tissue degradation by bacteria

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We present a basic model for the penetration of healthy tissue by bacteria. This model is given by a coupled system of a parabolic and an ordinary differential equation. We are particularly interested in the corresponding travelling wave problem, which is expected to yield the relevant invasion speed of an infection. We show the existence of travelling waves for a half-line of speeds and determine the minimum speed explicitly. Depending on the size of the 'degradation rate', which is the important model parameter, we observe a linear or a nonlinear selection principle for the minimal speed.

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Formation of singularities in the crystalline curvature flow

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We are interested in the weighted curvature flow with a driving term. The structure of the driving term is suggested by the physics of the mono-crystal growth from vapor. We show existence and uniqueness of solution for initial data which are a perturbation of the Wulff shape, i.e. the equilibrium configuration. We show what kind of shape emerges throughout the evolution.

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On the long time behavior of some singular phase change models

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In this talk we present a survey on some recent results obtained in collaboration with several coauthors and addressing well posedness and long time behavior of some systems of PDE's related to the so-called Penrose-Fife model for phase transitions. These systems are characterized by free energies containing singular terms, which are often difficult to control especially in the framework of a long-time analysis. We discuss some techniques which permit to get existence both of global attractors and of nonempty omega-limits for this kind of systems. We believe that some of the methods could be successfully applied also to other systems of PDE's with singular terms.

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Simulation analysis of Liesegang-like precipitation patterns

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In 1896, R. E. Liesegang discovered ordered patterns of precipitation which is well known as the Liesegang bands (rings). Since then, many researchers in the fields of chemistry, physics, mathematics and other scientific fields over 100 years have been intensively investigated to reveal the mechanism of such precipitation. Recently, Toramaru and his colleague have discovered a new transition phenomenon of precipitation patterns between periodic ones and tree-like ones. In order to understand such transition phenomenon, we introduce a new type of reaction diffusion model which is formulated in the phase-field setting.

In this talk, I would like to show that the model enables to explain phenomenologically the mechanism of such transition.

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Discrete Precipitation in a Reaction-Diffusion System

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Some reaction-diffusion systems show discrete spatial regions where precipitated material is found. This phenomenon has first been observed by Liesegang around 1890. He noticed in particular a regular spatial distribution of (what are now called) Liesegang bands/rings. In this lecture, we discuss a model, proposed by Keller and Rubinov (1981). We restrict ourselves to the case of one space dimension and we show that, given appropriate conditions, this model implies the existence of infinitely many discrete Liesegang bands. There still remain several open problems. This is joint work with D. Hilhorst, M. Mimura and I. Ohnishi.

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