Special Session 33: Nonlinear Elliptic and Parabolic Problems

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The goal is to cover recent developments in nonlinear elliptic and parabolic problems including:
- equations related to differential geometry
- higher order equations in conformal geometry
- free boundary problems
- topological and geometrical methods
- degenerate diffusion equations
- functional analytical approaches, maximal regularity
- computer assisted proofs.

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On boundedness of solutions of reaction-diffusion equations with nonlinear boundary conditions

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We give conditions on the nonlinearities of a reaction-diffusion equation with nonlinear boundary conditions that guarantee that any solution starting at a bounded initial data is bounded locally around certain point x_0 of the boundary, uniformly for all positive time. The conditions imposed are of local nature and need only to hold in a small neighborhood of the point x_0.

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Nodal solutions of a semiclassical nonlinear Schrödinger equation

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In the talk we are concerned with solutions $u \in H^1(R^N)$ of the equation $-\varepsilon^2\Delta u + a(x)u = |u|^{p-2}u$ where $2 < p < 2N/(N-2)$. We are interested in 2-nodal solutions, i.e. solutions which have precisely two nodal domains. We present new multiplicity results for 2-nodal solutions as $\varepsilon \to 0$. The proof combines variational ideas with dynamical systems methods and has several new features. Unlike other papers on this equation we do not need penalization techniques nor finite-dimensional reductions. This allows us to obtain even infinitely many solutions for certain potential functions a.

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Positivity preserving property for a class of biharmonic elliptic problems

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The lack of a general maximum principle for biharmonic equations suggests to study under which boundary conditions the positivity preserving property holds. We show that this property holds in general domains for suitable linear combinations of Dirichlet and Navier boundary conditions. These boundary conditions are also of some interest in semilinear equations, since they enable us to give explicit radial singular solutions to fourth order Gelfand-type problems.

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Strongly Competing Species in Special Domains

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Veronica Felli

We deal with strongly competing multispecies systems of Lotka-Volterra type with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u_i = f_i(x,u_i) - \varkappa u_i \sum_{j\neq i} u_j, & \text{in } \Omega, \\ \quad u_i = 0, & \text{on } \partial\Omega^n, \end{cases}$$

for $i = 1,\dots,k$. We show that the shape of the spatial domain can contribute to the occurrence of pattern formation (coexistence) and prevents extinction of all the species as the competition $\varkappa$ grows indefinitely. As a result we

prove the existence of complete solutions for a remarkable system of variational inequalities involved in segregation phenomena and optimal partition problems.

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Existence, uniqueness and approximation of a doubly degenerate nonlinear parabolic system

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We consider the following nonlinear parabolic system

$$\frac{\partial u}{\partial t} - c\Delta u = -f(u)\,v \quad \text{in } \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^d;$$
$$\frac{\partial v}{\partial t} - \nabla \cdot (b(u)\,\nabla[\psi(v)]\,) = \theta f(u)\,v \quad \text{in } \Omega_T$$

subject to no flux boundary conditions, and non-negative initial data u^0 and v^0 on u and v. Here we assume that $c > 0$, $\theta \geq 0$ and that $f \in C^2([0,\infty))$ is increasing with $f(0) = 0$. The system is possibly doubly-degenerate in that $b \in C^2([0,\infty))$ is only non-negative, and $\psi \in C^2([0,\infty))$ is convex, strictly increasing with $\psi(0) = 0$ and possibly $\psi'(0) = 0$. The above models the spatiotemporal evolution of a bacterium on a thin film of nutrient, where u is the nutrient concentration and v is the bacterial cell density. Under some further mild technical assumptions on b and ψ, we prove the existence and uniqueness of a weak solution to the above system. Moreover, we prove error bounds for a fully practical finite element approximation of this system.

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Parabolic boundary value problems with inhomogeneous symbols

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Classical parabolic theory of boundary value problems is based on the Fourier transform and the symbols of differential operators. Here the homogeneity of the symbols leads to solvability results and uniform a priori-estimates in corresponding Sobolev spaces.

In several applications in Mathematical Physics, however, there exists no homogeneous symbol of the operators, and classical parabolic theory does not apply. An example for this is the Stefan problem, a free boundary value problem. It is possible to develop a concept of parabolic boundary problems which includes these examples and gives solvability results and a priori estimates for them. This concept is based on the so-called Newton polygon related to the boundary value problem and the Lopatinskii matrix.

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Four manifolds with constant fourth order curvature

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In this talk we will present some new results on the existence of a conformal metric with constant Q-curvature (the curvature associated to a fourth order conformally covariant operator acting on four manifolds, the Paneitz operator). These results generalize previous results by Alice Chang and Paul Yang (”Extremal metrics of zeta function determinants on 4-manifolds”, Annals of Math. - 1995). This is a joint work with Andrea Malchiodi (SISSA-Italy).

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Existence and multiplicity results for semilinear equations with measure data

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In this paper, we study existence and nonexistence of solutions for the Dirichlet problem associated to the equation $-\Delta u = g(x,u) + \mu$ where μ is a Radon measure. Existence and nonexistence of solutions strictly depend on the nonlinearity $g(x,u)$ and suitable growth restrictions are assumed on it. Our proofs are obtained by standard arguments from critical theory and in order to find solutions of the equation, suitable functionals are introduced by mean of approximation arguments and iterative schemes.

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Existence of radial solutions for the p-Laplacian elliptic equations with weights

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Some existence results are obtained both for crossing radial solutions and for positive or compactly supported radial ground states of

$$\operatorname{div}(g(|x|)|Du|^{p-2}Du) + h(|x|)f(u) = 0 \text{ in } \mathbb{R}^n \setminus \{0\},$$

where $p > 1$, $n \geq 1$, g, $h \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ and $f \in C(\mathbb{R}^+) \cap L^1(0,1)$ is possibly singular at $u = 0$. When $f < 0$ near $u = 0$ we establish existence of ground states, while if f is positive we obtain existence of crossing solutions, using the definition of solution and qualitative properties established in a paper by Pucci, Garcìa-Huidobro, Manàsevich and Serrin (2006).

The proof technique is based on previous papers of Tang (1999) and of Gazzola, Serrin and Tang (2000) and on a new subcritical condition on f at infinity, introduced by Acciaio and Pucci (2003). Furthermore we obtain a non existence theorem for radial ground states, adapting a technique developed by Ni and Serrin (1985).

Then, applying the existence result of compactly supported radial ground states, together with an existence theorem of dead cores, contained in a recent paper by Pucci and Serrin (2006), we prove the existence of solutions, which involve both a dead core and *bursts* within the core, when the symmetric domain is sufficiently large.

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Triple junctions in geometric evolution equations: Analysis and computations

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John Barrett and Robert Nürnberg

In the talk we will discuss second and fourth order geometric evolution equations on networks including triple and quadruple junctions. We will briefly discuss stability results for stationary solutions. The main part of the talk will be used to present a flexible new numerical method to compute solutions to geometric evolution equations - possible applications include surface diffusion, (inverse) mean curvature flow and Willmore flow. We discuss stability estimates for the discretized problem and show that the discretization has very good properties with respect to the tangential equidristibution of mesh points. The talk concludes with several computational examples.

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Radial entire solutions for supercritical biharmonic equations

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We prove existence and uniqueness (up to rescaling) of positive radial entire solutions of supercritical semilinear biharmonic equations. The proof is performed with a shooting method which uses the value of the second derivative at the origin as a parameter. This method also enables us to find finite time blow up solutions. Finally, we study the convergence at infinity of smooth solutions towards the explicitly known singular solution. It turns out that the convergence is different in space dimensions $n \leq 12$ and $n \geq 13$.

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On the stochastic thin-film equation

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In this talk, we will be concerned with the effects thermal fluctuations have on the wetting behaviour of thin liquid films. Starting from incompressible Navier-Stokes equations with noise, we use long-wave-approximation and Fokker-Planck-type arguments to derive a fourth-order degenerate parabolic stochastic partial differential equation with multiplicative noise inside a convective term – the stochastic thin-film equation. We discuss the existence of a.s. non-negative solutions and give both formal and numerical evidence for our conjecture that thermal fluctuations may resolve discrepancies with respect to time-scales of dewetting between physical experiments and deterministic numerical simulations. This is joint work, partially with K.Mecke and M.Rauscher, partially with N.Dirr.

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Global solutions for superlinear parabolic equations involving the biharmonic operator for initial data with optimal slow decay

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We are interested in stability/instability of the zero steady state of the superlinear parabolic equation $u_t + \Delta^2 u =$

$|u|^{p-1}u$ in $\mathbb{R}^n \times [0,\infty)$, where the exponent is considered in the "super-Fujita" range $p > 1+4/n$. We determine the corresponding limiting growth at infinity for the initial data giving rise to global bounded solutions. In the supercritical case $p > (n+4)/(n-4)$ this is related to the asymptotic behaviour of positive steady states, which the authors have recently studied. Moreover, it is shown that the solutions found for the parabolic problem decay to 0 at rate $t^{-1/(p-1)}$.

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Critical Elliptic Systems in Potential Form

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We discuss various results concerning critical elliptic systems in potential form. We insist on blow-up type results, as the asymptotic analysis of sequences of solutions and/or compactness of sequences of solutions.

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Quaslinear Parabolic Systems with Mixed Boundary Conditions on Nonsmooth Domains

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In this talk we consider quasilinear systems of reaction-diffusion type with mixed Dirichlet-Neumann conditions on non smooth domains. Using techniques from maximal regularity and heat kernel estimates we prove existence and uniqueness to systems of this type in the L^p-setting.

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Lotka-Volterra type cross-diffusion models

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This talk presents some results for Lotka-Volterra type models in the presence of cross-diffusion effects. It discusses the existence and the non-existence of non-homogeneous steady state solutions and deals with the existence of traveling wave solutions in the presence of cross-diffusion effects.

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On surfaces with prescribed mean curvature and partially free boundaries

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We discuss continuous, stationary surfaces with prescribed mean curvature and partially free boundaries $\{\Gamma, S\}$ in Euclidean 3-space. At first, we present a new result concerning regularity at the free boundary for a smooth support surface S. Then we will study the behaviour near meeting points of the Jordan arc Γ with S and near "edge type" singular points of S itself. These results generalize G. Dziuk's investigations on minimal surfaces.

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On some differential inequalities

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We present some recent results for a wide class of differential inequalities including differential equations and variational inequalities. First we show some results about multiplicity and asymptotic behaviour of solutions in bounded domain of R^N near resonance, extending previous results by Ambrosetti, Rabinowitz, Crandall and Wang, also giving a nonlinear version of Courant's Nodal Theorem. In the second part we will be concerned with comparison results for solutions of differential inequalities on complete Riemannian manifolds, extending analogous results in domains of R^N by Pucci and Serrin.

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The flow of a heavy fluid past fixed obstacles: linear and non linear problems

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We discuss the problem of the steady two-dimensional flow past fixed disturbances in an open channel of finite depth. We consider different types of obstacles like submerged or surface-piercing bodies and localized perturbations of a horizontal bottom, and describe recent results and open questions on unique solvability of the

linear problem. The implications for the rigorous proof of the non linear, free boundary problem are discussed.

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Enclosure Methods for Elliptic Partial Differential Equations

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The lecture will be concerned with numerical enclosure methods for nonlinear elliptic boundary value problems. Here, analytical and numerical methods are combined to prove rigorously the existence of a solution in some "close" neighborhood of an approximate solution computed by numerical means. Thus, besides the existence proof, verified bounds for the error (i.e. the difference between exact and approximate solution) are provided.
For the first step, consisting of the computation of an approximate solution ω in some appropriate Sobolev space, no error control is needed, so a wide range of well-established numerical methods (including multigrid schemes) is at hand here. Using ω, the given problem is rewritten as a *fixed-point equation* for the error, and the goal is to apply a *fixed-point theorem* providing the desired error bound.
The conditions required by the chosen fixed-point theorem (e.g., compactness or contractivity, inclusion properties for a suitable subset etc.) are now verified by a combination of analytical arguments (e.g. explicit Sobolev embeddings, variational characterizations etc.) and verified computations of certain auxiliary terms, in particular of eigenvalue bounds for the linearization of the given problem at ω.
The method is illustrated by several examples (on bounded as well as on unbounded domains), where in particular it gives existence proofs in cases where no purely analytical proof is known.

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A-priori bounds for semilinear elliptic equations in Lipschitz domains

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The topic of this talk are a-priori bounds for positive solutions of semilinear elliptic equations with zero Dirichlet boundary values on bounded Lipschitz domains. The nonlinearity $f(x,s)$ is superlinear and grows at most like s^p for some $p>1$. After a short survey on known a-priori bounds on smooth bounded domains, I will focus on a recent result of Souplet and Quittner, where they show that an old critical exponent $p_{BT}=\frac{n+1}{n-1}$ of Brezis and Turner is sharp in the sense that for $p<p_{BT}$ a-priori bounds hold whereas for $p>p_{BT}$ there exist unbounded very weak solutions. I will explain how one can obtain similar sharp critical exponents on certain Lipschitz domains with corners, which generalize the Brezis-Turner exponent. For finite-difference discretizations of nonlinear Laplace equations we have similar results.

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Quantization issues for fourth order elliptic equations in dimension four

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In dimension four, the fourth order elliptic nonlinear equation $\Delta^2 u = e^u (E)$ enjoys conformal invariance properties that allow blowing up of sequences of solutions to (E). This invariance corresponds to a similar one for second order equations in dimension two studied, among others by Brézis-Merle: they actually enlightened a quantization of the energy associated to the dimension two. Surprisingly, there is absolutely no such a quantization in dimension four, and the situation can become quite weird. In this talk, we will describe completely the blow-up in the general case and in interesting specific situations.

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A priori Bounds for Positive Solutions of Semilinear Elliptic Systems

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We establish a priori bounds for positive solutions of semilinear elliptic systems of the form

$$(S)\qquad \begin{cases} -\Delta u = g(x,v)\ ,\ \text{in } \Omega \\ -\Delta v = f(x,u)\ ,\ \text{in } \Omega \\ u>0\ ,\ v>0\ \text{ in }\ \Omega \\ u=v\ =\ 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is a bounded and smooth domain in $\Re^2$. The nonlinearities $f(x,t)$ and $g(x,t)$ are superliner functions in t, and have at most exponential growth. This work extends some results of H. Brezis and F. Merle to semilinear systems.

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The uniformization method for quasilinear elliptic equations

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We shall present a new approach to the Dirichlet problem for the nonparametric equation of prescribed mean curvature H=H(x,y,z) in Euclidean space. Here we introduce isothermal parameters into the associated first fundamental form and obtain F.Rellich's nonlinear elliptic system. The latter has the Laplacian as its principal part and grows on the right-hand side quadratically in the gradient. With the aid of pseudoholomorphic functions, we estimate their solutions in an adequate way - initiated by E.Heinz. Via Banach's fixed point theorem, we locally solve the Dirichlet problem above and attain the global result by a nonlinear continuity method.

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Regularization of outflow problems in unsaturated porous media

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We investigate equations describing porous media with saturated, unsaturated, and dry regions. Due to a degenerate permeability coefficient $k = k(x,s)$ and a degenerate capillary pressure function $p = p_c(x,s)$, the equations may be of elliptic, parabolic, or of ODE-type. We construct a parabolic regularization of the equations and find conditions that guarantee the convergence of the parabolic solutions to a solution of the degenerate system. An example shows that the convergence fails in general. Our approach provides an existence result for the outflow problem in the case of x-dependent coefficients, and a method for a numerical approximation.

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Positive solutions for quasilinear elliptic equations with weights

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We study a quasilinear elliptic equation with weights and we prove the existence of positive radial ground states by using the Mountain Pass theorem and the constrained minimization method. We also give some non-existence results and some qualitative properties of the solutions.

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Heat equation with dynamical boundary conditions of locally reactive type

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Juan Luis Vazquez

We study the well–posedness of the initial boundary problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty)\times\Omega, \\ u_\nu = 0 & \text{on } (0,\infty)\times\Gamma_0, \\ u_t = -k_1 u_\nu & \text{on } (0,\infty)\times\Gamma_1, \\ u_t = k_2 u_\nu & \text{on } (0,\infty)\times\Gamma_2, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded regular open domain in $\mathbb{R}^N$ $(N \geq 1)$, $\Gamma = \partial\Omega$, ν is the outward normal to Ω, $k_1, k_2 > 0$ and $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_i, $i = 0,1,2$ are pairwise disjoint, possibly empty, measurable subsets of Γ with respect to Lebesgue surface measure on Γ. The main novelty lies on the reactive dynamical boundary condition imposed on Γ_2. The technique allows to study the more general initial–boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } Q = (0,\infty)\times\Omega, \\ u_\nu = \sigma(x) u_t & \text{on } [0,\infty)\times\Gamma, \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is as before and $\sigma \in L^\infty(\Gamma)$. A key step in our analysis consists in studying the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \sigma\Delta u = u_\nu & \text{on } \Gamma. \end{cases}$$

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On weakly harmonic maps from Finsler to Riemannian manifolds

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Sven Winklmann

We discuss weakly harmonic maps from Finsler manifolds into Riemannian manifolds and prove interior Hoelder estimates for such mappings whose image is contained in a regular ball of the target manifold. These estimates generalize results of Giaquinta, Hildebrandt, and Hildebrandt, Jost, Widman for weakly harmonic maps between Riemannian manifolds. As an application we obtain a Liouville theorem for entire harmonic maps from Finsler manifolds.

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Enclosures for variational inequalities

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We present a new method for proving the existence of a unique solution of variational inequalities within guaranteed close error bounds to a numerical approximation. The method is derived for a specific model problem featuring most of the difficulties of perfect plasticity. For this problem we summarize the existence properties and we present regularity results for the dual solution including estimates for the regularity at the boundary if the nonlinearity is restricted to the interior of the domain. Then we introduce a finite element method for the computation of admissible primal and dual solutions which guarantees the unique existence of a dual solution (by the verification of the safe load condition) and which allows for the determination of a guaranteed error bound. Moreover, we discuss conditions for the existence of a unique primal solution. Finally, we present explicit existence results and error bounds in some significant configurations illustrating the derived a priori finite element error estimate as well as the obtained a posteriori bounds.

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