# Special Session 39: Hemivariational Inequalities, Nonsmooth and Nonconvex Variational Problems with Applications

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The mathematical theory of hemivariational inequalities and their applications in mechanics, engineering and economics, were initiated and developed by P.D. Panagiotopoulos. This theory may be considered as an extension of theory of variational inequalities studied by G. Fichera, J.L. Lions and G. Stampacchia. Today research on hemivariational inequalities is considered as a part of applied nonlinear analysis devoted to the study of inequality problems.

Main topics of the session include hemivariational inequalities, nonlinear analysis, variational problems, nonsmooth problems, differential equations, nonconvex analysis, and various applications. The purpose of this session is to get together experts on these topics, to exchange scientific information on analysis, modeling, dynamics, computations and applications, and to discuss recent topics in theoretical and applied aspects.

The session celebrates the sixty fifth birthday of Z. Denkowski.

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### Optimal control for impulsive systems on the space of finitely additive measures

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We consider optimal control problems for evolution equations and inclusions on the space of finitely additive measures $\mathcal{M}_{ba}(E)$ where $E$ is a Banach space. The evolution equations are described in the weak form given by

$$\begin{aligned} d\mu_t(\varphi) &= \mu_t(\mathcal{A}\varphi)dt + \mu_t(\mathcal{B}_u\varphi)dt \\ &\quad + \mu_{t-}(\mathcal{C}(t)\varphi)\nu(dt),\ t \geq s \\ \mu_s(\varphi) &= \pi(\varphi),\ \forall\ \varphi \in \mathcal{F} \end{aligned}$$

where $\mathcal{F} \subset B(E)$ denotes the class of bounded Borel measurable functions on $E$ having Frechet derivatives with bounded supports. The operators $\mathcal{A}$, $\mathcal{B}_u$ and $\mathcal{C}$ are given by

$$\begin{aligned} \mathcal{A}\varphi(\xi) &\equiv\ < A^* D\varphi(\xi), \xi >_{E^*,E} \\ &\quad + < D\varphi(\xi), f_0(t,\xi) >, \\ \mathcal{B}_u\varphi(t,\xi) &\equiv\ < D\varphi(\xi), f(t,\xi,u) > \\ \mathcal{C}\varphi(t,\xi) &\equiv \int_0^1 < D\varphi(\xi + rg(t,\xi)\nu(\{t\})), \\ &\quad g(t,\xi) >_{E^*,E} dr. \end{aligned}$$

This class of systems arises naturally from the controlled differential equations of the form

$$dx = Axdt + f_0(t,x)dt + f(t,x,u)dt + g(t,x(t-))\nu(dt),$$

($t \geq 0$) with non smooth, unbounded, measurable vector fields $f_0, f, g$; unbounded linear operator $A$ generating a $C_0$-semigroup on $E$; and $\nu$ a signed measure and $u$ an admissible control. We consider several control problems for this system and min-max games for uncertain dynamic systems described by inclusions.

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### Comparison Results for a Class of Quasilinear Evolutionary Hemivariational Inequalities

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We consider a class of quasilinear evolutionary hemivariational inequalities under nonmonotone multivalued flux boundary conditions. Our main goal is to provide existence and comparison results in terms of appropriately defined sub- and supersolutions on the basis of which we then prove compactness and extremality results of the solution set within some sector.

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### Degree theoretic methods in the study of positive solutions for nonlinear hemivariational inequalities

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In this paper we study the existence of positive solutions for nonlinear elliptic problems driven by the $p$-Laplacian differential operator and with a nonsmooth potential (hemivariational inequalities). The hypotheses, in the case $p = 2$ (semilinear problems), incorporate in our framework of analysis the so-called asymptotically linear problems. The approach is degree theoretic based on the fixed point index for nonconvex-valued multifunctions due to Bader.

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**Complete Solutions to a Class of Nonconvex/Nonsmooth Variational Problems**

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Nonconvex/nonsmooth problems appear naturally in many applications, such as chaotic dynamics, post-buckling of thin-walled structures, phase transitions of modern materials, and certain biological processes like DNA dynamics, etc. Due to nonconvexity and nonsmoothness of the total potential energy concerned, traditional analysis and related numerical methods for solving these problems have proven to be very difficult, or even impossible. In global optimization and computational science, many nonconvex/nonsmooth problems are NP-hard. In nonconvex dynamical systems, numerical methods may produce the so-called chaotic solutions. However, the canonical duality theory developed from nonconvex mechanics can be used to solve these problems.

In this talk, the speaker will present a brief review and some new developments on the canonical duality theory with applications to a class of variational problems in nonconvex mechanics and global optimization. These nonconvex problems are directly related to a large class of semi-linear partial differential equations in mathematical physics including phase transitions, post-buckling of large deformed beam model, chaotic dynamics, nonlinear field theory, and superconductivity. Numerical discretizations of these equations lead to a class of very difficult global minimization problems in finite dimensional space. However, by the use of the canonical dual transformation, these nonconvex constrained primal problems can be converted into certain very simple canonical dual problems. The criticality condition leads to dual algebraic equations which can be solved completely. Therefore, a complete set of solutions to these very difficult primal problems can be obtained. The extremality of these solutions are controlled by the so-called triality theory. Several examples will be illustrated by movies.

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**Existence Results for Quasilinear Hemivariational Inequalities at Resonance**

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We consider quasilinear hemivariational inequality at resonance. We obtain two existence theorems using a Landesman-Lazer type condition. The method of the proof is based on the nonsmooth critical point theory for locally Lipschitz functions.

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**New Applications of the Method of Moreau and Panagiotopoulos**

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Electrical devices are described in terms of Ampere-Volt characteristics. Experimental measures and mathematical models show that the ampere-volt characteristics of various devices like diodes and thyristors are set-valued graphs involving vertical branches.

In this paper, we will show that the approach of Moreau and Panagiotopoulos can be extended so as to constitute a powerful mathematical tool that can be used to write a precise and rigorous mathematical model describing the dynamics of circuits involving devices like diodes, Zener diodes, varactors, thyristors, diacs and silicon controlled rectifiers.

We will give in a rigorous and precise way the steps one has to follow in order to get a mathematical model that can be used to describe the dynamics of a circuit involving such devices.

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**A Class of Evolution Hemivariational Inequalities for Dynamic Piezoelectric Contact Problems with Friction**

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We consider a mathematical model of a dynamic frictional contact between a piezoelectric body and an obstacle. The constitutive relations of piezoelectricity are assumed. The contact is modeled by boundary conditions which are nonmonotone, possibly multivalued and are expressed by the Clarke subdifferential of locally Lipschitz functions. We derive a variational formulation of the model which is of the form of a coupled system of a hyperbolic hemivariational inequality and an elliptic equation. We provide conditions under which the system involving as unknowns the displacement field and the electric potential has a solution.

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### Non-resonance and Resonance for Hemivariational Inequalities

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Consider the following nonlinear elliptic eigenvalue problem

$$\begin{cases} -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) - \lambda|x(z)|^{p-2}x(z) \\ \qquad \in \partial j(z,x(z)) \text{ a.e. on } Z \\ x|_{\partial Z} = 0,\ 1 < p < +\infty. \end{cases}$$

Here $Z \subset R^N$ is a bounded domain with a $C^{1,\alpha}$-boundary $\partial Z$ $(0 < \alpha < 1)$, $j(z,x)$ is a measurable function of $(z,x) \in Z \times R$ which is locally Lipschitz in the $x$ variable and $\partial j(z,x)$ stands for the generalized subdifferential of $x \mapsto j(z,x)$ in the sense of Clarke. This problem belongs to the class of eigenvalue problems for hemivariational inequalities. We focus on the existence of positive solutions and nontrivial multiple solutions to the eigenvalue problem under different assumptions. Denoting by $\lambda_1$ the principal eigenvalue of the negative $p$-Laplacian $(-\Delta_p, W_0^{1,p}(Z))$ we show the existence of positive solutions in the non-resonant case $\lambda < \lambda_1$ and in the resonant case $\lambda = \lambda_1$. Knowing that in general we should not expect having positive solutions in the case of near resonance from the right, i.e., $\lambda > \lambda_1$ close to $\lambda_1$, we supply in this situation existence and multiplicity results for nontrivial solutions. Our results extend in a nonsmooth quasilinear framework several classical properties of single- or multi-valued semilinear Dirichlet boundary value problems at non-resonance, resonance and near resonance. This exposition follows the paper:

D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, Positive Solutions and Multiple Solutions at Non-resonance, Resonance and Near Resonance for Hemivariational Inequalities with $p$-Laplacian, submitted.

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### On a General Equilibrium Model in Reflexive Banach Space

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The motivation for this paper is the Walrasian general equilibrium model of economy, as formulated by K. J. Arrow and G. Debreu. The problem considered takes the form of a system of variational inequalities on a reflexive Banach space as the infinite dimensional commodity space. The conditions sufficient for existence of solutions are provided by means of theory of pseudomonotone multivalued mapping due to F. E. Browder and P. Hess and Fenchel duality theory combined with Galerkin method. The analysis is carried out without any lattice considerations. The substantial difference of presented approach in comparison with currently applied methods is that the commodity space is not required to have interior points, the preferences need not satisfy any variant of $\omega$-properness assumption and the consumption sets are not required to have a cone structure. The paper affords new existence results for both finite and infinite dimensional setting.

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### Existence and stability of solutions to semilinear wave equation with Dirichlet boundary control

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The paper is devoted to the study existence and stability of weak solutions for the following hyperbolic equations with controls in Dirichlet boundary conditions:

$$x_{tt}(t,z) - \Delta_z x(t,z) = f(t,z,x(t,z)) \text{ on } (0,T)\times\Omega \tag{1}$$

$$x(0,z) = \varphi(0,z),\ x_t(0,z) = \psi(0,z) \text{ on } \Omega \tag{2}$$

$$x(t,z) = v(t,z) \text{ on } (0,T)\times\Gamma \tag{3}$$

$$v(t,z) \in \mathbf{V} \text{ on } (0,T)\times\Gamma \tag{4}$$

where $\Omega$ is a given bounded domain of $R^n$ with boundary $\Gamma = \partial\Omega$ of $C^2$, $\Sigma = (0,T)\times\Gamma$, $\mathbf{V}$ is a given nonempty, closed set in $R$, $f : [0,T]\times\bar{\Omega}\times R\times R^m \to R$, and $\varphi,\psi : R^{n+1} \to R$ are given functions, $\varphi(0,\cdot) \in L^2(\Omega)$, $\psi(0,\cdot) \in H^{-1}(\Omega)$; $x : [0,T]\times\Omega \to R$, $(x;x_t) \in C([0;T];L^2(\Omega)) \times C([0;T];H^{-1}(\Omega))$ and $v : (0,T)\times\Gamma \to R^m$ belongs to $L^2(\Sigma)$. In [1] the existence and regularity problems for the above problem was study in a case of $f$ independent on $x$ with $f$ belonging to $L^1(0,T;H^{-1}(\Omega))$. We assume that the function $f$ is nonlinear in $x$ and under some specific assumption on it we prove that solution of our equation $(x;x_t) \in C([0;T];L^2(\Omega)) \times C([0;T];H^{-1}(\Omega))$. Next under additional assumption if we take a sequence $L^2(\Sigma) \ni v_k \to v_0$ then we show that the corresponding sequence $x_k$

of solutions to (1)-(4) is convergent in some defined sense to solution $x_0$ of (1)-(4) with $v_0$ in (3).

References:
[1] I. Lasiecka, J. L. Lions and R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl. 65 (1986), 149–192.

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**A class of hemivariational inequalities for viscoelastic materials with long-term memory**

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We consider a class of abstract evolution hemivariational inequalities in the study of frictional contact problems for viscoelastic materials with long-term memory. In the model dynamic equation of motion is considered with the viscoelastic constitutive relationship of the Kelvin-Voigt type, the contact is bilateral and the friction is modeled with the Tresca law. The term responsible for memory of the body is of the integral form with a linear continuous operator. The multivalued boundary condition comes from the nonconvex superpotential and is formulayed as an inclusion with a general subdifferential. The latter leads to a hemivariational inequality as a variational formulation of our problem. The aim of this paper is to establish the existence of weak solutions to the problem by using arguments of evolution hemivariational inequalities, in particular a surjectivity result for multivalued pseudomonotone mappings.

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**Multiple positive solutions and sign-changing solutions**

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We use degree theoretic and variational methods to study the existence of multiple positive solutions and of sign changing solutions for nonlinear elliptic equations driven by the $p$-Laplacian.

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**On the regularization of sliding modes**

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Approximability of sliding motions for control systems governed by nonlinear finite-dimensional differential equations is considered. This regularity property is shown to be equivalent to Tikhonov well-posedness of a related minimization problem in the context of relaxed controls. This allows us to give a general approximability result, which in the autonomous case has an easy to verify geometrical formulation. Then we consider non-approximable sliding mode control systems. In the flavour of regularization of ill-posed problems, we propose a method of selection of well-behaved approximating trajectories converging to a prescribed ideal sliding.

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**Generalizations of the Lax-Milgram theorem**

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We prove a linear and a nonlinear generalization of the Lax-Milgram theorem. In particular we give sufficient conditions for a real-valued function defined on the product of a reflexive Banach space and a normed space to represent all bounded linear functionals of the latter. We also give two applications to singular differential equations.

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