Special Session 9: Formation and Dynamics of Patterns in Evolution Equations

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One of the most interesting features of spatially explicit nonlinear systems is the existence of complicated patterns. These can either be stationary patterns that act as organizing centers for the dynamics, or patterns that change subtly or dramatically with time. These types of patterns are easily seen through experimental observation or numerical simulation, and are often generated by nonlinear parabolic partial differential equations. In this special session results related to the formation and the dynamics of patterns will be presented, from a variety of different points of view. These include patterns arising in concrete applications, stochastic mechanisms for the formation of patterns, the creation of complicated stationary patterns in parameter-dependent models, as well as descriptions of their dynamics.

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Weakly nonlinear asymptotics of the $\kappa - \theta$ model of cellular flames

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M. Frankel, J. Hulshof, A. Lunardi and G.I. Sivashinsky

We consider the $\kappa - \theta$ model of flame front dynamics introduced by Frankel, Gordon and Sivashinsky. We show that a periodic problem for the latter system of two equations is globally well-posed. We prove that near the instability threshold the front is arbitrarily close to the solution of the Kuramoto-Sivashinsky equation on a fixed time interval if the evolution starts from close configurations. We present numerical simulations that illustrate the theoretical results and also demonstrate the ability of the $\kappa - \theta$ to generate chaotic cellular dynamics.

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Nucleation in the one-dimensional Cahn-Hilliard model.

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To describe the phenomenon of nucleation we consider the one-dimensional Cahn-Hilliard-Cook equation

$$\partial_t u + \frac{1}{\lambda^2}\partial_x^4 u = \partial_x^2(f(u)) + \sigma\xi$$

with homogeneous initial condition $u_0 = \mu$ and stochastic noise term ξ, where μ is in the metastable region. For noise intensity $\sigma = 0$ the initial condition u_0 is a stable equilibrium. However, for $\sigma > 0$ the trajectories can overcome the attracting influence of the underlying deterministic system and leave the domain of attraction D almost surely. We are interested in whether there is a set occupying only a small portion of the boundary ∂D where trajectories leave D with high probability. In the deterministic model we use a reduction to an inertial manifold and continuity arguments in the parameter λ to prove the existence of heteroclinic orbits from the first spikes to u_0 respectively. These spikes are the unique minimizers of the free Ginzburg-Landau energy on the boundary ∂D. With suitable noise and by extending methods due to Freidlin and Wentzell we can then prove that for sufficiently small noise intensity $\sigma > 0$ the trajectories leave D with overwhelming probability close to the first spike solutions.

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On the asymptotic behaviour of nonlocal phase separation processes

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In our talk we present a model of nonlocal phase separation processes in multicomponent systems of particles. These processes are driven by the minimization of the free energy under the constraint of mass conservation. The free energy functional contains both a convex logarithmic part describing the FERMI-type behaviour of the particles and a nonconvex quadratic part taking into account nonlocal particle interaction. This leads to an evolution system of second order parabolic equations for the mass densities including nonlinear drift terms.

The assumptions on the interaction operator, which ensure the unique solvability and the regularity of the problem in suitable function spaces, are quite general. The key quantity to study the asymptotic behaviour is the free energy, which turns out to be a LYAPUNOV functional for the system. Using the regularity of the solution, and a ŁOJASIEWICZ–SIMON gradient inequality for the free energy, we get strong convergence results for the whole trajectory to a stationary point. At the end of the talk we show some numerical simulations to illustrate our analytic results.

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Stabilisation in a bistable dispersal equation

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In this talk I will discuss the dynamics of the integrodifferential equation

$$u(x,t)_t = \rho(\int_\Omega J(x,y)u(y,t)dy - u(x,t) + f(u(x,t)),$$

where Ω is a bounded domain, $J(x,y)$ is a suitable nonnegative kernel and $f(u)$ is a bistable nonlinearity. I will concentrate on the conditions under which one can prove stabilisation to equilibria, by methods that rely on showing precompactness of orbits and those that exploit the structure of the nonlinearity in a perturbation argument.

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The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system

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We consider the Allen-Cahn equation $u_t = \Delta u + \varepsilon^{-2} f^\varepsilon(x,t,u)$, where ε is a small parameter and f^ε a bistable nonlinearity, which is associated with a double-well potential whose well-depth is slightly unbalanced by order ε and we consider rather general initial data u_0 that is independent of ε. We perform a rigorous analysis of both generation and motion of interface. More precisely we show that the solution develops a steep transition layer within the timescale of order $\varepsilon^2|\ln\varepsilon|$, and that the layer starts to move afterwards in a much slower time scale.

We then prove that the thickness of the transition layer is of order ε and, moreover, that the limit interface (for $\varepsilon = 0$) approximates the actual interface (for $\varepsilon > 0$) within an error margin of order ε. This is an optimal estimate that has not been known before, even in the case that the double-well potential is balanced.

Finally we extend our results to systems of reaction-diffusion equations which include the FitzHugh-Nagumo system as a special case.

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Similarity solutions involving boundary value problems: The example of free convection and some more general case

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In this talk we will first focus on a boundary value problem involving a third order non-linear autonomous differential equation. This problem arises when looking for similarity solutions for free convection boundary-layer flows along a semi-infinite vertical permeable surface embedded in a fluid saturated porous medium. We will show how to obtain results about the existence, nonexistence and uniqueness of these solutions using a plane dynamical system deduced from the equation and give the long time behaviour of the unbounded solutions. Then we will present some results about the concave and convex solutions for a more general problem that can be applied to the previous situation as well as to a wide range of physical problems including mixed convection, liquid metals in magnetic fields, the steady MHD equation and the Falkner-Skan equation.

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Stability analysis of phase boundary motion by surface diffusion in a bounded domain

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Harald Garcke and Kazuo Ito

The motion of the phase boundary by the geometrical evolution law in a bounded domain $\Omega \subset \mathbb{R}^2$ is studied in this talk. We consider the surface diffusion flow equation $V = -m\gamma\kappa_{ss}$ which is the gradient flow of the length functional of the evolving curves $\{\Gamma_t\}_{t>0}$ with respect to the H^{-1}-inner product. Here V and κ are the normal velocity and the curvature of the evolving curve Γ_t, respectively. s is the arc-length parameter along Γ_t, $m > 0$ is the mobility constant, and $\gamma > 0$ is the constant concerning the surface energy of Γ_t.

Our goal in this talk is to obtain the nonlinear stability of the stationary solutions for the two-phase problem by using the criteria of the linearized stability of stationary solutions, which are derived by investigating the sign of eigenvalues for the corresponding eigenvalue problem. Also, the criteria of the linearized stability of stationary solutions for the three-phase problem with triple junction will be showed in some special cases.

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Cylinder Buckling and a Mountain Pass Solution

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We revisit the classical problem of the buckling of a long thin axially compressed cylindrical shell. Typically this will buckle into localized solutions that may be viewed as homoclinics. A key question is to determine at what load this occurs. By examining the energy landscape of the perfect cylinder we deduce an estimate of the sensitivity of the shell to imperfections. Key to obtaining this is the existence of a mountain pass point for the system. Numerically the mountain pass solution with lowest energy has the form of a single dimple. We interpret these results and validate the lower bound against some experimental results available in the literature.

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Random Field Kac Model: From micro to macro structures.

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Materials exhibiting a fine mixture of phases are familiar and well studied in many settings. Such mixtures can arise for a variety of reasons. I focus here on those materials where the presence of the impurities causes the microscopic structure to vary from point to point and my attempt is to underlying contiguity with different areas, like variational calculus, PDE, probability and Statistical Mechanics. Essential point in the whole approach is the use of both microscopic and macroscopic description in capturing the physical phenomena. I will recall some results on equilibrium and non equilibrium statistical mechanics setting for Kac Ising spin system with random external field and their relation to variational continuum models and PDE.

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The Lojasiewicz inequality in the pattern formation

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We are interested in the asymptotic behavior of solutions to differential equations, this is understood as convergence to a steady state, traveling wave or self-similar solution. We present the gradient Lojasiewicz inequality and its generalizations. Further, we show how one can used it to deduce convergence to a steady state solutions to gradient and gradient-like flows.

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Spinodal decomposition on general domains

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Thomas Wanner and Jonathan Desi

I will discuss results on spinodal decomposition in the Cahn-Hilliard model on general domains. Several previous results have shown that starting near a homogeneous equilibrium on a rectangular domain, solutions to the linearized and the nonlinear Cahn-Hilliard equation behave indistinguishably up to large distances from the homogeneous state. I will show how these results can be extended to the unit disk, for which interesting new phenomena can be observed. These are the first results of this kind for domains more general than rectangular. I will also discuss the case of general domains and its relationship to a conjecture of Aurich et al., regarding the asymptotic behavior of the infinity-norms of the eigenfunctions of the Laplacian. The proof is based on vector-valued extensions of probabilistic methods used by Wanner.

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Evolution of support in multidimensional thin-film flow with nonlinear diffusion, convection and absorption

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In a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \leq 3)$ we consider the initial-boundary Neumann problem for the fourth order degenerate quasilinear parabolic equation:

$$u_t + \mathrm{div}\,(\mathrm{a_0|u|^n\nabla\Delta u - a_1|u|^m\nabla u})$$

$$+\sum_{i=1}^{N} b_i(|u|^{\lambda_i - 1}u)_{x_i} + b_0|u|^{\lambda-1}u = 0,$$

$$a_0 > 0,\ a_1 \geq 0\ b_0 \geq 0,\ \lambda > 0,$$

$$\lambda_i > 0,\ 0 < n < 3,\ m \geq 0,\ b_i \in \mathbb{R}^1$$

with initial data $u_0 \in H^1(\Omega),\ \mathrm{supp\,u_0} \Subset \Omega$. Under suitable conditions on the parameters in the equation we prove finite speed propagation of the support for arbitrary generalized (entropy) solutions $u(x,t)$ and obtain sharp upper estimates on this propagation with dependence on the direction of convection (“slow” and “fast” interfaces). We investigate starting time for the motion of interfaces with dependence on the local behavior of $u_0(x)$ in the vicinity of the boundary of its support: flatness conditions guaranteeing appearance of a waiting time for the spreading of the support, conditions for backward motion of the interface.

The methodology of the above mentioned analysis of the propagation properties of energy solutions of higher order quasilinear parabolic equations of monotone type [1], is based on the study of properties of solutions of various “homogeneous” and “unhomogeneous” functional inequalities and systems of such inequalities (Stampacchia lemma like statements).

[1] A. Shishkov, A. Shchelkov. Izvestiya: Mathematics 62:3 (1998), 601–626.

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Closed orbits on non-compact hypersurfaces

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C. Viterbo proved that any $(2n-1)$-dimensional compact hypersurface $M \subset (\mathbb{R}^{2n}, \omega)$ of contact type has at least one closed characteristic. This result proved the Weinstein conjecture for the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. Various extensions of this theorem have been proved since, all for compact hypersurfaces. In this talk we consider *non-compact* hypersurfaces $M \subset (\mathbb{R}^{2n}, \omega)$ coming from mechanical Hamiltonians, and prove an analogue of Viterbo's result. The main result provides a strong connection between the homology groups $H_k(M)$, $k = n, \cdots, 2n-1$, and the existence of closed characteristics in the non-compact case (including the compact case).

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Oscillating solutions in autonomous parabolic PDE

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It is a common feature of ”almost all” parabolic PDE that ”almost all” of their solutions exhibit one of the following two types of behavior: Either (1) they approach infinity (in some appropriate topology) in finite or infinite time, or (2) they are global in time and converge to some stationary pattern. Many affirmative results in this direction are known, especially for second-order equations. The talk concentrates on some examples in which case (3) occurs: Namely, we present some autonomous second-order parabolic problems which allow for oscillating global solutions. Both bounded and unbounded solutions are constructed which do not stabilize asymptotically, neither to a bounded nor to an unbounded pattern.

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