# Special Session 16: Reaction Diffusion Equations and Applications

**Jerome Goddard**, Auburn University, USA
**Ratnasingham Shivaji**, University of North Carolina at Greensboro, USA

Recent developments in reaction diffusion equations have greatly increased their importance and usefulness in modeling physical and biological phenomena in many disciplines. The application of reaction diffusion is seemingly endless with their use naturally arising in areas such as biology, ecology, chemistry, geology, physics, and engineering. Investigation of the structure of steady states for such models yields interesting nonlinear elliptic boundary value problems of varied types. Even with the study of elliptic boundary value problems having such a rich mathematical history dating back to the 1960s, much is still unknown about the structure of solutions to such problems. This session will facilitate the exploration of current applications of reaction diffusion, proof techniques, and open questions of nonlinear elliptic boundary value problems.

### Existence of solutions to boundary value problems at full resonance

**Kristen Abernathy**
Winthrop University, USA
abernathyk@winthrop.edu
**Jesus Rodriguez**

The focus of this talk is the study of nonlinear differential equations of the form

$$\dot{x_i}(t) = a_i(t)x_i(t) + f_i(\epsilon, t, x_1(t), \cdots, x_n(t)),$$

with $i = 1, 2, \cdots, n$, subject to two-point boundary conditions

$$b_i x_i(0) + d_i x_i(1) = 0,$$

for $i = 1, 2, \cdots, n$. We formulate sufficient conditions for the existence of solutions based on the dimension of the solution space of the corresponding linear, homogeneous equation and the properties of the nonlinear term when $\epsilon = 0$. We focus on the case when the solution space of the corresponding linear, homogeneous equation is $n$-dimensional; that is, when the system is at full resonance. The argument we use relies on the Lyapunov-Schmidt Procedure and the Schauder Fixed Point Theorem.

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### On the solvability of nonlinear Sturm-Liouville problems

**Zachary Abernathy**
Winthrop University, USA
abernathyz@winthrop.edu
**Jesus Rodriguez**

In this talk, we establish sufficient conditions for the existence of solutions to the nonlinear differential equation

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t))$$

subject to general non-local boundary conditions of the form

$$\begin{cases} \alpha x(0) + \beta x'(0) + \eta_1(x) = \phi_1(x) \\ \gamma x(1) + \delta x'(1) + \eta_2(x) = \phi_2(x). \end{cases}$$

We will emphasize the relationship between the eigenvalues of a related linear Sturm-Liouville problem and the rate of growth of nonlinearities present in both the differential equation and boundary conditions. This relationship will then motivate explorations of singular Sturm-Liouville problems.

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### Existence of alternate steady states in a phosphorous cycling model

**Dagny Butler**
Mississippi State University, USA
dg301@msstate.edu
**Sarath Sasi, Ratnasingham Shivaji**

We analyze the positive solutions to the steady state reaction diffusion equation with Dirichlet boundary conditions of the form:

$$\begin{cases} -\Delta u = \lambda[K - u + c\dfrac{u^4}{1+u^4}], \quad x \in \Omega \\ \quad u = 0, \quad x \in \partial\Omega. \end{cases}$$

Here $\Delta u = div\big(\nabla u\big)$ is the Laplacian of $u$, $\frac{1}{\lambda}$ is the diffusion coefficient, $K$ and $c$ are positive constants, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in $C^2$. This model describes the steady states of phosphorus cycling in stratified lakes. Also, it describes the colonization of barren soils in drylands by vegetation. In this paper, we discuss the existence of multiple positive solutions leading to the occurrence of an S-shaped bifurcation curve. We prove our results by the method of sub-super solutions.

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### Evolution of dispersal and the ideal free distribution

**Robert stephen Cantrell**
University of Miami, USA
rsc@math.miami.edu
**Chris Cosner, Yuan Lou**

A general question in the study of the evolution of dispersal is what kind of dispersal strategies can convey competitive advantages and thus will evolve. We consider a two species competition model in which the species are assumed to have the same population dynamics but different dispersal strategies. Both species disperse by random diffusion and advection along certain gradients, with the same random dispersal rates but different advection coefficients. We find a conditional dispersal strategy which results in the ideal free distribution of species, and show that it is a locally evolutionarily stable strategy. We further show that this strategy is also a globally convergent stable strategy under suitable assumptions, and our results illustrate how the evolution of dispersal can lead to an ideal free distribution. The underlying biological reason is that the species with this particular dispersal strategy can perfectly match the environmental resource, which leads to its fitness being equilibrated across the habitat.

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**Existence and nonexistence of positive solutions for a special class of elliptic systems**

**Maya Chhetri**
UNC Greensboro, USA
maya@uncg.edu

We consider an elliptic system of the form

$$\begin{cases} -\Delta u - \mu\Delta v = f(v) & \text{in} \quad \Omega \\ -\Delta v - \lambda\Delta u = g(u) & \text{in} \quad \Omega \\ \quad u = 0 = v & \text{on} \quad \partial\Omega\,, \end{cases}$$

where $\lambda, \mu > 0$ are parameters, $\Omega$ is a bounded domain in $R^N$ with smooth boundary $\partial\Omega$. The nonlinearities $f, g : [0, \infty) \to R$ are $C^1$ functions that are sublinear at infinity. We discuss existence and nonexistence of positive solutions. We consider both positone and semipositone reaction terms in our analysis.

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**Population models with diffusion, strong Allee effect, and nonlinear boundary conditions**

**Jerome Goddard**
Auburn University, USA
jgoddard@aum.edu
**E. Lee, R. Shivaji**

We discuss the steady state solutions of a diffusive population model with strong Allee effect. In particular, this study is focused on a population that satisfies a certain nonlinear boundary condition and on its persistence when constant yield harvesting is introduced. We prove the existence of at least two positive steady states of the model for certain parameter ranges. These existence results are established by the method of sub-super solutions.

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**A reaction-diffusion problem with nonlocal reaction**

**Georg Hetzer**
Auburn University, USA
hetzege@auburn.edu

Incorporating a bio-feedback into an energy balance climate model leads to a nonautonomous functional reaction-diffusion problem with a set-valued reaction term which depends on a nonlocal Volterra operator. A global existence result for nonnegative solutions and the existence of a trajectory attractor are discussed.

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**An existence result for an infinite semipositone problem**

**Lakshmi Kalappattil**
Mississippi State University, USA
lk154@msstate.edu
**Jerome Goddard, Eun Kyoung Lee, R. Shivaji**

We consider the singular boundary value problem:

$$\begin{cases} -\Delta u = \dfrac{au - bu^2 - c}{u^\alpha}, & x \in \Omega \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\Delta$ is the Laplacian operator, $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$ and $a, b, c, \alpha$ are positive constants where $\alpha$ is in $(0, 1)$. Let $\lambda_1$ be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. When $a > \lambda_1$, we prove there exists a $c^*(a, b, \alpha, \Omega)$ such that if $c < c^*$, the problem has a positive solution. We prove the existence result by the method of sub-super solutions. We also extend our results to the case when $\Omega$ is an exterior domain.

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**Analysis of class of elliptic equations with nonlinear boundary condtions arising in combustion theory**

**Eunkyung Ko**
Mississippi State University, USA
ek94@msstate.edu
**Peter Gordon, R. Shivaji**

We study positive solutions to the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ \mathbf{n} \cdot \nabla u + C(u)u = 0, & x \in \partial\Omega, \end{cases}$$

where $C : [0, \infty) \to (0, \infty)$ is a $C^1$ non decreasing function, $\lambda > 0, \Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1$ and $f : [0, \infty) \to (0, \infty)$ is a $C^1$ non decreasing function such that $\lim_{u\to\infty} \frac{f(u)}{u} = 0$. We establish the existence of a positive solution for all $\lambda > 0$, and discuss the existence of multiple positive solutions and uniqueness results for certain ranges of $\lambda$ when $f$ satisfies certain additional assumptions. A simple model that satisfies all our hypotheses is $f(u) = \exp[\frac{\alpha u}{\alpha+u}]$ for $\alpha \gg 1$. We prove our existence and multiplicity results by the method of sub-supersolutions, and our uniqueness result by establishing apriori estimates.

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**Existence of the second positive solution for a p-Laplacian problem**

**Eun Kyoung Lee**
Pusan National University, Korea
eunkyoung165@gmail.com

We investigate the existence, nonexistence, and multiplicity of positive radial solutions for the p-Laplacian problem with boundary parameters. For proofs, we mainly use a combination of a fixed point theorem, the method of upper and lower solutions in the frame of the ordinary differential equations (ODE) technique. This is a joint work with Chan-Gyun Kim and Yong-Hoon Lee.

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**A balanced finite element method for singularly perturbed reaction-diffusion problems**

**Runchang Lin**
Texas A&M International University, USA
rlin@tamiu.edu
**Martin Stynes**

Consider the singularly perturbed linear reaction-diffusion problem $-\varepsilon^2\Delta u + bu = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$, where $d \geq 1$, the domain $\Omega$ is bounded with (when $d \geq 2$) Lipschitz-continuous boundary $\partial\Omega$, and the parameter $\varepsilon$ satisfies $0 < \varepsilon \ll 1$. It is argued that for this type of problem, the standard energy norm $v \mapsto [\varepsilon^2|v|_1^2 + \|v\|_0^2]^{1/2}$ is too weak a norm to measure adequately the errors in solutions computed by finite element methods: the multiplier $\varepsilon^2$ gives an unbalanced norm whose different components have different orders of magnitude. A balanced and stronger norm is introduced, then for $d \geq 2$ a mixed finite element method is constructed whose solution is quasi-optimal in this new norm. By a duality argument it is shown that this solution attains a higher order of convergence in the $L_2$ norm. Error bounds derived from these analyses are presented for the cases $d = 2, 3$. For a problem posed on the unit square in $\mathbb{R}^2$, an error bound that is uniform in $\varepsilon$ is proved when the new method is implemented on a Shishkin mesh. Numerical results are presented to show the superiority of the new method over the standard mixed finite element method on the same mesh for this singularly perturbed problem.

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**Strong bounded solutions for nonlinear parabolic equations**

**Nsoki Mavinga**
Swarthmore College, USA
mavinga@swarthmore.edu
**M. Nkashama**

We are concerned with the existence of bounded solutions existing for all times for nonlinear parabolic equations with nonlinear boundary conditions on a domain that is bounded in space and unbounded in time (the entire real line). We establish a priori estimates for solutions to linear boundary value problems, and derive a weak maximum principle which is valid on the entire real line in time. We then use comparison techniques, a priori estimates, and nonlinear approximation methods to prove the existence and, in some instances, positivity and uniqueness of bounded solutions existing for all times.

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**A study of delayed cooperation diffusion system with Dirichlet boundary conditions**

**Abdur Raheem**
indian Institute of Technology, Kanpur, India
araheem@iitk.ac.in
**Dhirendra Bahuguna**

In this paper we study the existence and uniqueness of strong solution of a delayed cooperation diffusion system with Dirichlet boundary conditions using the method of semi discretization in time.

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**Asymptotic behavior of the solutions of the BVP governing Marangoni convection**

**Susmita Sadhu**
Southwest Minnesota State University, USA
susmita.sadhu@smsu.edu
**J. Paullet**

The boundary value problem (BVP) governing the Marangoni convection over a flat surface is given by

$$f''' = \frac{2k+1}{3}f'^2 - \frac{k+2}{3}ff''$$
$$f(0) = 0, \quad f''(0) = -1, \quad f'(\infty) = 0,$$

where $k > -1$ is the temperature gradient exponent. It has been proved by J. Paullet that for each $k \in (-1, -1/2)$, the BVP admits a continuum of solutions. In this talk, we will consider the asymptotics of these solutions. We will prove that for each $k \in (-1, -1/2)$, there exists a solution $f_0$ of the BVP that satisfies

$$f_0'(\eta) \sim c_0 f_0(\eta)^{-\frac{3(k+1)}{k+2}} \exp\left(-\int_{\eta_0}^{\eta} f_0(s)\, ds\right)$$

as $\eta \to \infty$ for some $\eta_0 > 0$ sufficiently large, and a constant $c_0 > 0$ that depends only on $k$ and $\eta_0$. We conjecture that the BVP has exactly one solution that obeys the above asymptotics, i.e. its derivative decays to zero exponentially, while the derivatives of the other solutions decay to zero algebraically.

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**Alternate stable states in ecological systems**

**Sarath Sasi**
Mississippi State University, USA
ss885@msstate.edu

We consider the existence of multiple positive solutions to the steady state reaction diffusion equation with Dirichlet boundary conditions of the form:

$$\begin{cases} -\Delta u = \lambda[u - \dfrac{u^2}{K} - c\dfrac{u^2}{1+u^2}], \quad x \in \Omega, \\ \quad u = 0, \quad x \in \partial\Omega. \end{cases}$$

Here $\Delta u = div(\nabla u)$ is the Laplacian of $u$, $\frac{1}{\lambda}$ is the diffusion coefficient, $K$ and $c$ are positive constants and $\Omega \subset \mathbb{R}^N$ is a smooth bounded region with $\partial\Omega$ in $C^2$. This model describes the steady states of a logistic growth model with grazing in a spatially homogeneous ecosystem. It also describes the dynamics of the fish population with natural predation. In this talk we discuss the existence of multiple positive solutions leading to the occurrence of an S-shaped bifurcation curve. We also introduce a constant yield harvesting term to this model and discuss the existence of positive solutions including the occurrence of a $\Sigma$-shaped bifurcation curve in the case of a one-dimensional model. We prove our results by the method of sub-super solutions and quadrature method.

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**On radial solutions of polyharmonic equations with power nonlinearities**

**Paul Schmidt**
Auburn University, USA
pgs@auburn.edu
**Monica Lazzo**

A vast amount of literature on second-order semilinear elliptic equations with power-like nonlinearities is concerned with the existence, uniqueness or multiplicity, and qualitative behavior of solutions to various types of boundary-value problems. Lacking a maximum principle, higher-order analogues of such problems require entirely new methods, even if expected results are similar to what is known in the second-order case. As a first step towards a better understanding of the higher-order case, we have been using dynamical-systems methods to study radially symmetric solutions of polyharmonic equations with pure power nonlinearities. We give a survey of our results and their implications.

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**Spatiotemporal mutualistic model of mistletoes and birds**

**Junping Shi**
College of William and Mary, USA
jxshix@wm.edu
**Chuncheng Wang, Rongsong Liu, Carlos Martinez del Rio**

A mathematical model which incorporates the spatial dispersal and interaction dynamics of mistletoes and birds is derived and studied to gain insights of the spatial heterogeneity in abundance of mistletoes. Fickian diffusion and chemotaxis are used to model the random movement of birds and the aggregation of birds due to the attraction of mistletoes respectively. The spread of mistletoes by birds is expressed by a convolution integral with a dispersal kernel. Two different types of kernel functions are used to study the model, one is Dirac delta function which reflects one extreme case that the spread behavior is local, and the other one is a general non-negative symmetric function which describes the nonlocal spread of mistletoes. When the kernel function is taken as the Dirac delta function, the threshold condition for the existence of mistletoes is given and explored in term of parameters. For the general non-negative symmetric kernel case, we prove the existence and stability of non-constant equilibrium solutions. Numerical simulations are conducted by taking specific forms of kernel functions. Our study shows that the spatial heterogeneous patterns of the mistletoes are related to the specific dispersal pattern of the birds which carry mistletoe seeds.

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**Infinite semipositone problems with asymptotically linear growth forcing terms**

**Ratnasingham Shivaji**
University of North Carolina at Greensboro, USA
shivaji@uncg.edu
**Hai Dang, Lakxhmi Shankar**

We study positive solutions to the singular problem

$$\begin{cases} -\Delta u = \lambda f(u) - \dfrac{1}{u^\alpha} \quad \text{in } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda$ is a positive parameter, $\Omega$ is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$, 0

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**Existence of solutions for degenerate elliptic $p(x)$-Laplacian**

**Inbo Sim**
University of Ulsan, Korea
ibsim@ulsan.ac.kr
**Yun-Ho Kim**

We study the following nonlinear problem

$$-\mathrm{div}(w(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) \quad \text{in } \Omega$$

which is subject to Dirichlet boundary condition. Under suitable conditions on $w$ and $f$, employing the variational method, we show the existence of solutions for the above problem in the weighted variable exponent Lebesgue-Sobolev spaces. Also we show the positivity of the infimum eigenvalue for the problem. This is a joint work with Yun-Ho Kim.

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**Dynamical behaviour of spatio-temporal plankton population model**

**Shilpee Srivastava**
Indian Institute of Technology Delhi, India
shilpee2007@rediffmail.com

We investigate the dynamical behaviour of three dimensional plankton population model using analytical and numerical techniques, modified by the addition of diffusive terms to represent the effect of random motion. A comparative study of local stability in the presence and absence of diffusion has been performed.

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