Special Session 29: Self-Organized Behavior of Nonlinear Elliptic Equations and Pattern Formation of Strongly Interacting Systems

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Several phenomena can be described by a number of densities (of mass, population, probability, ...) distributed in a domain and subject to diffusion, reaction, and either cooperative or competitive interaction. Whenever the interaction is the prevailing mechanism, one can reasonably expect pattern formation and, in the competitive case, that the several densities can not coexist and tend to segregate hence determining a partition of the domain.

We focus on the analysis of the qualitative properties of solutions of systems of semilinear elliptic equations, whenever the interaction is not negligible with respect to diffusion (e.g. when the parameter describing the competition diverges to infinity). Prototype of variational systems are those of Gross-Pitaevski equations, describing the motion of a Bose-Einstein condensated quantum gas with different spin states. Strong competition appears in association with diverging interspecific scattering lengths. The limiting behaviour is known for the ground state stationary solutions and solitary waves: the wave amplitudes segregate, that is, their supports tend to be disjoint (phase separation). Of course, the partition becomes the main object of investigation both from the analytical point of view than from the geometric, with emphasis on the multiple points of multiple intersection. When the system possesses a variational structure, one can associate an optimal partition problem with the ground states. Conversely, one can regard at optimal partitions related to linear or nonlinear eigenvalues as limits of competing systems as the competition parameter diverges.

A key role is played by the entire solutions to systems, with polynomial interactions. In the case of an eigenvalue, the variational properties of the nodal partition associated with an eigenfunction is deeply connected to the number of nodal domains it is possible to link the minimizing property of the nodal partition to that of being sharp with respect to the Courant's nodal Theorem.

On a prescribed mean curvature equation in modeling MEMS

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Mathematical models of the form $\mathfrak{M}u = f(u;\lambda)$, where $\mathfrak{M}$ is the nonparametric mean curvature operator and λ is a nonlinear eigenvalue, have been studied since the early 1800s, beginning with the work of Thomas Young and Pierre-Simon Laplace. Recently, it has been shown that when $f(u;\lambda) = \lambda(1+u)^{-2}$ the aforementioned model describes the shape of an electrostatically deflected thin, elastic membrane. In this talk, we look at the contrast between this particular prescribed mean curvature equation and its standard approximation $\Delta u = f(u)$.

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Solutions with point singularities for a MEMS equation with fringing field

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We construct solutions of the equation

$$-\Delta u = \frac{\lambda(1+|\nabla u|^2)}{(1-u)^2}, \quad 0$$

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Singularity of eigenfunctions at the junction of shrinking tubes

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Consider two domains connected by a thin tube: it can be shown that, generically, the mass of a given eigenfunction of the Dirichlet Laplacian concentrates in only one of them. The restriction to the other domain, when suitably normalized, develops a singularity at the junction of the tube, as the channel section tends to zero. Our main result states that, under a nondegeneracy condition, the normalized limiting profile has a singularity of order $N-1$, where N is the space dimension. We give a precise description of the asymptotic behavior of eigenfunctions at the singular junction, which provides us with some important information about its sign near the tunnel entrance. More precisely, the solution is shown to be one-sign in a neighborhood of the singular junction. In other words, we prove that the nodal set does not enter inside the channel.

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Solutions for a semilinear elliptic equations involving critical exponents.

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Let Ω be a bounded domain in $\mathbb{R}^N$. We consider the problem

$$-\Delta u = u^p + \lambda u^q, \quad u > 0 \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{in} \quad \partial\Omega,$$

where $p = (N+2)/(N-2)$. We describe large solutions for this problem, and we obtain precise estimates for $\lambda = \lambda(c)$ where $c := \max_{\Omega} u$ and tends to infinity. This depends strongly in the dimension N and the value of q. The case $q = 1$ is the well know Brezis-Nirenberg problem. We also study a characterization of the point in Ω where solution concentrates as $c \to \infty$.

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Hot-spot solutions in a model of crime

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We study a model of crime developed recently by Short and collaborators at UCLA. This model exhibits hot-spots of crime – localized areas of high criminal activity. In a certain asymptotic limit, we use singular perturbation theory to construct the profile of these hot-spots and then study their stability.

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Concentration behaviour in biharmonic equations of MEMS.

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A Micro electro-mechanical systems (MEMS) capacitor is a microscopic device consisting of two plates held opposite one other. The lower plate is immobile while the upper plate is fixed along its edges but free to deflect in the presence of an electric potential towards the lower plate. The deflecting upper plate may reach a stable equilibrium, however, if the applied potential exceeds a threshold, known as the pull-in voltage, the upper plate will touchdown on the lower plate. When certain physical assumptions are applied, the deflection of the upper surface can be modeled as a fourth order PDE with a singular non-linearity. It is shown that the model captures the pull-in instability of the device and provides a prediction of the pull-in voltage. When the pull-in voltage is exceeded, the equations develop a finite time singularity (FTS) and it is demonstrated that this FTS forms on multiple isolated points or on a continuous set of points. The touchdown set is predicted for certain domains by means of asymptotic expansions. We also illustrate a new model which regularizes the infinite electric field and allows the dynamics to be continued beyond the initial FTS.

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The effects of alliances in competing species models

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We analyze the effects of spatial refuges in heterogeneous competing species systems as well as the effects of strategic alliances in competitive environments. The spatial refuges under strong competition enhance segregation and, hence, diversity. The facilitative effects increase dramatically the stability of the ecosystems.

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Triple-junctions in a strong interaction limit of a three-component system

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We consider a three-component reaction-diffusion system with a reaction rate parameter, and investigate its singular limit as the reaction rate tends to infinity. The limit problem is described by a nonlinear cross-diffusion system. The system is regarded as a weak form of a free boundary problem which possesses three types of free boundaries. Triple junction points appear at the intersection of the three interfaces. Furthermore, the dynamics is governed by a system of equations in each region separated by the free boundaries.

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Convergence of minimax and continuation of critical points for singularly perturbed systems

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In the recent literature, the phenomenon of phase separation for binary mixtures of Bose–Einstein condensates can be understood, from a mathematical point of view, as governed by the asymptotic limit of the stationary Gross–Pitaevskii system

$$\begin{cases} -\Delta u + u^3 + \beta u v^2 = \lambda u \\ -\Delta v + v^3 + \beta u^2 v = \mu v \\ u, v \in H^1_0(\Omega), \quad u, v > 0, \end{cases}$$

as the interspecies scattering length β goes to $+\infty$. For this system we consider the associated energy functionals J_β, $\beta \in (0, +\infty)$, with L^2–mass constraints, which limit J_∞ (as $\beta \to +\infty$) is strongly irregular. For such functionals, we construct multiple critical points via a common minimax structure, and prove convergence of critical levels and optimal sets. Moreover we study the asymptotics of the critical points.

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Nonnegative solutions of elliptic equations and their nodal structure

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We consider the Dirichlet problem for a of class nonlinear elliptic equations on reflectionally symmetric bounded domains. We are mainly interested in nonnegative, nonzero solution which are not strictly positive. We will discuss the existence of such solutions and their symmetry properties, including the symmetry of their nodal set.

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A double bubble solution in a ternary system with long range interaction

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We consider a ternary system of three constituents, a model motivated by the triblock copolymer theory. The free energy of the system consists of two parts: an interfacial energy coming from the boundaries separating the three constituents, and a longer range interaction energy that functions as an inhibitor to limit micro domain growth. We show that a perturbed double bubble exists as a stable solution of the system. Each bubble is occupied by one constituent. The third constituent fills the complement of the double bubble. The location of the double bubble is determined by the Green's function of the Laplace operator on the sample domain.

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Optimal partition problems involving Laplacian eigenvalues

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Given a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, and a positive integer $m \in \mathbb{N}$, consider the following optimal partition problem:

$$\inf\{\sum_{i=1}^{m} \lambda_k(\omega_i) : \omega_i \subset \Omega \text{ open } \forall i, \\ \omega_i \cap \omega_j = \emptyset \text{ whenever } i \neq j\},$$

where $\lambda_k(\omega)$ denotes the k-th eigenvalue of $-\Delta$ in $H^1_0(\omega)$. Approximating this problem by a system of elliptic equations with competition terms, we show the existence of regular optimal partitions. Moreover, multiplicity of sign-changing solutions for the approximating system is obtained.

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Extremality conditions for optimal partitions

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We deal with the free boundary problem associated with optimal partitions related with linear and nonlinear eigenvalues. We are concerned with extremality conditions and the regularity of the interfaces. These properties are then linked with extremality conditions of the nodal set of eigenfunctions and the number of their nodal components.

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Natural constraints in variational methods and superlinear Schroedinger systems

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For a C^2-functional J defined on a Hilbert space X, we consider the set

$$\mathcal{N} = \{x \in A : \mathrm{proj}_{V_x} \nabla J(x) = 0\},$$

where $A \subset X$ is open and $V_x \subset X$ is a closed linear subspace, possibly depending on $x \in A$. We study sufficient conditions for a constrained critical point of J restricted to $\mathcal{N}$ to be a free critical point of J, providing a unified approach to different natural constraints known in the literature, such as the Birkhoff-Hestenes natural isoperimetric conditions and the Nehari manifold. As an application, we prove multiplicity of solutions to a class of superlinear Schroedinger systems on singularly perturbed domains.

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Qualitative properties and existence results for an nonlinear elliptic system

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We consider the following nonlinear ellitpic system

$$\Delta U = UV^2,\ \Delta V = VU^2, U, V > 0 \text{ in } R^n$$

We first prove the existence, asymmetry, stability and local uniqueness of the one-dimensional solution. Then we prove the two-dimensional De Giorgi Conjecture and Stability Conjecture, under some growth condition. Finally we construct solutions with any polynomial growth at $+\infty$.

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Spiky patterns in a consumer chain model

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We study a consumer chain model which is based on Schnakenberg type kinetics. This model is realistic in a predator-prey context if cooperation of predators is prevalent in the system. The system also serves as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir. In this model there is a middle predator feeding on the prey and a final predator feeding on the middle predator. This means that the middle predator plays a hybrid role: it acts as both predator and prey. We will consider two cases: (i) Final predator has smaller diffusivity than the rest. (ii) Middle predator has smaller diffusivity than the rest.

We will rigorously prove the existence of spiky patterns for this system. We will also study their stability properties.

The analytical results are confirmed by numerical computations. Biological applications are discussed.

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Monge-Ampere equations on exterior domains

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We consider the Monge-Ampere equation $\det(D^2u) = f$ where f is a positive continuous function in R^n and is a perturbation of a positive constant at infinity. First we prove that every globally defined solution is close to a parabola plus a logarithmic term in two dimensional spaces and is close to a parabola in higher dimensional spaces. Then we show that given any prescribed asymptotic behavior mentioned above, there exists a unique global solution corresponding to it. Finally we solve the exterior Dirichlet problem with given function on the boundary of a bounded convex set and a prescribed asymptotic behavior at infinity.

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Regularity results for boundary partition problems

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We analyze the regularity of solutions to boundary partition problems. In particular, we consider a system of elliptic equations arising as constrained minima of a certain energy functional, under the condition that the solutions have traces with disjoint support on some portion of the boundary. Thanks to the link between boundary dynamics and fractional diffusion processes, our results extend also to segregation problems involving fractional powers of the laplace operator.

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