Special Session 36: Stochastic Partial Differential Equations and their Optimal Control

Wilfried Grecksch, Martin-Luther-University, Germany

This special session should give a general overview as well about new tendencies in the field of stochastic partial differential equations (spdes) as about optimal control problems for spdes. The invited speakers will discuss issues related to numerics/computations, existence and uniqueness of solution processes for spdes and properties of spdes. Another main goal will be to discuss optimality conditions for optimal control problems with spdes and methods to approximate optimal controls. Especially, spdes and applications are discussed driven by fractional noise processes.

Portfolio optimization under partial with expert opinions

Gabih Abdelali
University of Caddi Ayyad, Morocco
a.gabih@uca.ma
Rüdiger Frey, Ralf Wunderlich

We investigate optimal portfolio strategies in a market with partial information on the drift. The drift is modelled by continuous Markov chaine with finitely many states which are not observable. Information on the drift is obtained from observations on the stocks prices. Moreover Expert Opinions in the form of signals at random discrete time points are included in the analysis. We derive the filtering equation for the return process and incorporate the filter into the state variables of the optimization problem. This problem is studied with dynamic programing method. In particular we propose a policy improvement method to obtain computable approximations of the optimal strategy. Numerical results are presented at the end.

$\longrightarrow \infty \diamond \infty \longleftarrow$

Diffusion in heterogeneous domains

Vo Anh
Queensland University of Technology, Australia
v.anh@qut.edu.au
J.M. Angulo, M.D. Ruiz-Medina

This work is motivated by the problem of saltwater intrusion in coastal aquifers. These underground aquifers can be a major source of water for irrigation, industrial and town water usage. Due to their proximity to the ocean, excessive demand for groundwater may result in saltwater intrusion with a substantial loss of agricultural land. It is therefore an essential task to develop suitable mathematical models and computational tools for visualisation and prediction of salinity diffusion in aquifers for scenario analysis and management planning. Due to the complex nature of the aquifers, with alternating layers of permeable sandstone and impermeable clay sediments, it is a challenge to understand the role of heterogeneity in the aquifer and in the physical law governing the saltwater flow. We will formulate the problem in the framework of fractional diffusion in heterogeneous domains. We will outline the existing theory of diffusion on fractals, then move on to a theory of diffusion on multifractals based on the RKHS approach. This latter theory yields a class of models for fractional diffusion with variable singularity order in heterogeneous domains. We will look at the wavelet transforms of these space-time random fields for their statistical estimation and interpolation.

$\longrightarrow \infty \diamond \infty \longleftarrow$

A filtering problem for a linear stochastic Schrödinger equation

Wilfried Grecksch
Martin-Luther-University, Germany
wilfried.grecksch@mathematik.uni-halle.de

Let (V, H, V^*) be rigged complex Hilbert spaces and let $A : V \to V^*$ be a linear coercitve operator. If we choose $(H^1(G), L^2(G), H^{-1}(G))$ and the Laplacian operator Δ in a bounded domain G with homogeneous Dirichlet boundary conditions then we get an example for A. All processes are defined on a fixed complete probability space.The state equation is given by

$$dX(t) = iAX(t)dt + g(t)dB(t),\ t \geq 0\ X(0) = X_0,$$

where the solution process is defined by the variational solution. The noise process B is a Gaussian process with values in a real separable Hilbert space K and its covariance operator is defined by $C(t,s)Q$ and C is a real function with

$$\frac{\partial^2 C}{\partial t \partial s} \in L^2([0,T]^2)$$

and Q is a trace class operator in K. The fractional Brownian motion, the subfractional Brownian motion and Liouville fractional Brownian motion are examples for B. The observation process Y takes values in a n dimensional subspace H_n of H and has the form

$$dY(t) = G(t)X(t)dt + f_1(t)dB^n(t) + f_2(t)dW(t)$$
$$Y(0) = 0.$$

The noise B^n is a n dimensional part of B and W is a Brownian motion with values in H_n. g, f_1, f_2 are deterministic Hilbert Schmidt operator valued functions and the stochastic differentials are defined in

the sense of *Ito*.
We derive explicit expressions for the optimal filter $E\{X(t)|(Y(s))_{s\in[0,t]}\}$.

$\longrightarrow \infty \diamond \infty \longleftarrow$

Vector calculus on fractals and applications

Michael Hinz
FSU Jena and Univ. of Connecticut, Germany
Michael.Hinz.1@uni-jena.de
Michael Roeckner, Alexander Teplyaev

If a fractal set carries a diffusion (respectively local regular Dirichlet form) the existing analysis on fractals suffices to study many second order PDE. However, until recently the notion of gradient respectively first order derivative had not been understood very well. In this talk we present some new developments concerning a vector calculus on fractals that is flexible enough to have nice applications in PDE and SPDE and to provide some of the basics for related control problems.

$\longrightarrow \infty \diamond \infty \longleftarrow$

An optimal control problem for a nonlinear controlled stochastic Schrödinger equation

Diana Keller
Martin-Luther-University Halle-Wittenberg, Germany
diana.keller@mathematik.uni-halle.de

First, we consider a nonlinear controlled Schrödinger problem with additive noise, which is given by

$$\begin{aligned} dX(t,x) = &-iAX(t,x)\,dt + i\,U(t,x)X(t,x)\,dt \\ &+i\lambda f(X(t,x))\,dt + i\,dW(t,x) \end{aligned}$$

for all $t\in[0,T]$ and all $x\in(0,1)$, where A is the one-dimensional negative Laplacian, $\lambda\in\mathbb{R}_+$, $f(X(t,x))$ is a twice continuously differentiable, Lipschitz-continuous, growth-bounded nonlinearity and $W(t,\cdot)$ is an $L^2[0,1]$-valued Q-Wiener process. The admissible control $U(t,x)$ is a stochastic process such that a unique variational solution $X(t,x)$ exists. For the sake of simplicity, we choose initial and homogeneous boundary conditions in the following way

$$X(0,x)=\varphi(x)\in H^1[0,1], \qquad \forall\, x\in[0,1],$$
$$\frac{\partial}{\partial x}X(t,x)\Big|_{x=0} = \frac{\partial}{\partial x}X(t,x)\Big|_{x=1} = 0, \ \ \forall\, t\in[0,T].$$

Existence, uniqueness and some smoothness properties of the variational solution are discussed. Moreover, we are interested in minimising the objective functional

$$\begin{aligned} J(U) = \alpha_0\, E\int_0^T |X(t,0)-f_0(t)|^2\, dt \\ +\alpha_1\, E\int_0^T |X(t,1)-f_1(t)|^2\, dt \end{aligned}$$

relative to the control $U(t,x)$. Here, $X(t,x)$ is the solution of the nonlinear stochastic Schrödinger problem belonging to the control, $\alpha_0,\alpha_1\in\mathbb{R}$ and $f_0(t), f_1(t)\in L^2[0,T]$ are given functions. The question of solvability of this control problem is treated as well.

$\longrightarrow \infty \diamond \infty \longleftarrow$

On random partial differential equations

Antje Mugler
Brandenburg University of Technology Cottbus, Germany
mugler@tu-cottbus.de

In addition to stochastic partial differential equations driven by some kind of white noise also partial differential equations with random data especially random coefficients play an important role in a number of applications in engineering. For example the modelling of complex systems like subsurface flows with random permeability can be mentioned here. Also the solution of such random partial differential equations is very often part of uncertainty quantification procedures in this context and has been of growing interest in the last years.

In this talk we present conditions for the existence of a unique weak solution to an illustrative model problem where the random coefficient can be strictly bounded away from zero and above by random variables only. This is an important issue from the application point of view but the classical deterministic approach to solve the problem does not carry over to this case. So we introduce an alternative stochastic Galerkin approach yielding a sequence of approximate solutions converging to the exact solution in the natural topology. Furthermore we present an example where the solutions of the traditional stochastic Galerkin approach, which is widely used in practice nowadays, fails to converge to the exact solution in the natural topology.

This talk is based on joint work with Hans-Jörg Starkloff (Westsächsische Hochschule Zwickau).

$\longrightarrow \infty \diamond \infty \longleftarrow$

Regularization of ordinary and partial differential equations by noise

Michael Roeckner
University of Bielefeld, Germany
roeckner@math.uni-bielefeld.de

It is well known that there are ordinary differential equations (ODE) which have no or many solutions for a given initial condition, but have a unique solution if one adds a sufficiently large noise. We shall first explain this type of "regularization by noise" on the level of the corresponding Fokker-Planck-Kolmogorov equations both for ODE in finite and infinite dimensional state spaces. Then we shall recall a concrete ODE in finite dimensions given by a merely p-integrable vectorfield which has a unique strong solution when perturbed by a Brownian noise. Finally, we shall present an analogous new result in infinite dimensions, which is applicable to stochastic partial differential equations.

⟶ ∞ ⋄ ∞ ⟵

A maximum principle for a distributed stochastic optimal control problem

Christian Roth
Martin-Luther-University Halle-Wittenberg, Germany
christian.roth@mathematik.uni-halle.de
W. Grecksch

We introduce a controlled parabolic Itô equation

$$\begin{aligned} X(t,x) &= X(0,x) + \int_0^t \alpha\Delta_x X(s,x)\,ds \\ &\quad + \int_0^t f(s,X(s,x),U(s))\,ds \\ &\quad + \int_0^t \Big(X(s,x)g(s),\, dB^h(s)\Big) \end{aligned}$$

for a certain class $\mathbb{A}$ of admissible controls, where the solution is defined in the sense of [2] . The optimal control problem consists of

$$\min\{J(U,X) : U \in \mathbb{A}\},$$

where

$$J(U,X) = E\Phi(X(t)) + E\int_0^t f(s,X(s),U(s))\,ds.$$

An optimality condition of maximum principle type is developed.

Literature

[1] F. Biagini, Y. Hu, B. Øksendal and A. Sulem, A stochastic maximum principle for processes driven by fractional Brownian motion, *Stochastic Processes Appl.*, **100(2002)**.
[2] C. Roth and D. Julitz, An infinite dimensional quasilinear evolution equation driven by an infinite dimensional Brownian motion, *Festschrift in Celebration of Prof. Dr. Wilfried Grecksch's* 60^{th} *Birthday.*

⟶ ∞ ⋄ ∞ ⟵

Kolmogorov equations for randomly forced fluids

Martin Sauer
TU Darmstadt, Germany
sauer@mathematik.tu-darmstadt.de

We study so called generalized Newtonian fluids arising as a more general model of the Navier-Stokes equations in mathematical fluid dynamics. A prominent example are the Power Law or Ladyzhenskaya fluids having a power law type structure with exponent p for the stress tensor. The main features of such equations besides the better description of non-Newtonian behaviour are global uniqueness results depending on the exponent p. Under stochastic forcing these equations have been discovered only recently and in this talk we will consider them perturbed by additive noise. We prove existence of invariant measures and study the associated Kolmogorov operator in L^q-spaces of this measure. Under conditions on the exponent p it is possible to establish a uniqueness result.

⟶ ∞ ⋄ ∞ ⟵

Generalized polynomial chaos expansion and the solution of random pdes

Hans-Joerg Starkloff
University of Applied Sciences Zwickau, Germany
hans.joerg.starkloff@fh-zwickau.de

Certain partial differential equations with random data, e.g., random coefficients or forcing terms, can be solved approximately using Hermite or generalized polynomial chaos expansions in connection with stochastic Galerkin methods. In the talk some basic problems related to the use of generalized polynomial chaos expansions for such applications are investigated.

⟶ ∞ ⋄ ∞ ⟵

Random attractors for multivalued lattice dynamical systems with multiplicative noise

José Valero
Universidad Miguel Hernandez, Spain
jvalero@umh.es
T. Caraballo, F. Morillas

In this work we study a stochastic lattice dynamical system with multiplicative noise from the point of view of the theory of global random attractors. Using a suitable change of variable the stochastic system is transformed into a random one. The nonlinear term of the system satisfy some dissipative and growth conditions ensuring existence of solutions of the Cauchy problem, but no uniqueness. Hence, we define for it a multivalued random dynamical system and prove the existence of a random global attractor.

The more difficult part in the proof comparing with the single-valued case is to obtain the measurability of the attractor.

$\longrightarrow \infty \diamond \infty \longleftarrow$

Ergodic properties of stochastic curve shortening flow

Max Von Renesse
Univ. Leipzig, Germany
renesse@lmu.de
Abdelhadi Es-Sarhir, Wilhelm Stannat

We discuss 1+1 dimensional random interface models known as stochastic curve shortening flow. Well-posedness is established in a variational SPDE framework. For the long time behaviour we prove ergodicity using the lower-bound-technique by Peszat/Szarek/Komorowski. Finally we show polynomial stabilty using a-priori estimates on the invariant measure.

$\longrightarrow \infty \diamond \infty \longleftarrow$

Schrödinger equation with noise on the boundary

Frank Wusterhausen
Martin Luther University Halle-Wittenberg, Germany
frank.wusterhausen@mathematik.uni-halle.de

We are interested in existence and uniqueness of distributional solutions for the Schrödinger equation with Neumann noise, i.e.

$$\begin{aligned} i\partial_t u(t,y) - \Delta_y u(t,y) &= 0, & y \in G, t \in [0,T], \\ u(0,y) &= 0, & y \in G, \\ \partial_\nu u(t,y) &= \dot{W}(t,y), & y \in \partial G, t \in [0,T], \end{aligned}$$

where G is a bounded domain in $\mathbb{R}^n$ with smooth enough boundary ∂G and $\dot{W}$ is an L_2-valued white noise (white in time), i.e. $W(t,y) = \sum_{j=1}^{\infty} \gamma_j h_j(y) B^j(t)$, where $\gamma \in l_2$, $h_k \in L_2(\partial G)$ with $\|h_k\|_{L_2(\partial G)} = 1$ for all $k \in \mathbb{N}$, and $B^j(t)$, $t \in [0,T]$, $j = 1, \ldots, \infty$, denote independent one-dimensional standardized Brownian motions. The last equation is understood in the Itô sense.

A function (or distribution) u is called a distributional solution of the problem above if

$$i\int_0^T (u(t,\omega)|\dot{\varphi}(t))_{L_2(G)} dt + \int_0^T (u(t,\omega)|\Delta\varphi(t))_{L_2(G)} dt = -\int_0^T \sum_{j=1}^{\infty} \gamma_j (h_j|\varphi(t))_{L_2(\partial G)} dB_t^j(\omega)$$

holds for a.e. $\omega \in \Omega$ and for all $\varphi \in M$, where

$$\begin{aligned} M := \{&\varphi \in C^\infty([0,T] \times G) : \\ &\varphi(T,y) = 0 \ \forall y \in G, \\ &\partial_\nu \varphi(t,y) = 0 \ \forall t \in [0,T] \ \forall y \in \partial G\}, \end{aligned}$$

$(\cdot|\cdot)_{L_2}$ is the inner product in L_2, and $\dot{\varphi}$ denotes the derivative of φ with respect to t. We show that the problem admits a unique distributional solution

$$u \in C^\beta(0,T; L_2(\Omega; H_2^{-\alpha}(G)))$$

with $\forall \alpha \in (1/2, 3/2)\ \forall \beta \in (0, 1/2): \ \alpha - 2\beta > 1/2.$

For the proof we make use of spectral decomposition of the Laplacian with homogeneous Neumann boundary condition.

If we replace the Neumann boundary condition by Dirichlet boundary condition we have the following result:

$$u \in C^\beta(0,T; L_2(\Omega; H^{-\alpha}(G)))$$

with $\forall \alpha \in (3/2, 2)\ \forall \beta \in (0, 1/4): \ \alpha - 2\beta > 3/2.$

$\longrightarrow \infty \diamond \infty \longleftarrow$