# Special Session 39: Polynomial Methods for Differential Equations and Dynamical Systems

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**James Stanley Sochacki**, James Madison University, USA
**Roger Thelwell**, JMU, USA
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Since the advent of Taylor Series, polynomial methods have been used to solve differential equations and learn the properties of differential equations. There are many polynomial methods for solving differential equations and understanding dynamical systems. For example, Taylor polynomials, Chebyshev Polynomials and Adomian Polynomials are used to generate approximate solutions. Automatic differentiation can be used with power series and Cauchy products to generate solutions to differential equations. Pade Methods are based on ratios of polynomial functions and products of polynomials and are used to understand singularities in differential equations. Picard Iteration can be used with polynomial differential equations to obtain error estimates and convergence rates. This session intends to bring together researchers in these areas to develop a community and to share ideas and develop new tools and theory for understanding differential equations and dynamical systems and their relationship with polynomials.

### Sparse moment sequences

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The well-known theorems of Stieltjes, Hamburger and Hausdorff establish conditions on infinite sequences of real numbers to be moment sequences. Further, works by Carathéodory, Schur and Nevanlinna connect moment problems to problems in function theory and functions belonging to various spaces. In many problems associated with realization of a signal or an image, data may be corrupted or missing. Reconstruction of a function from moment sequences with missing terms is an interesting problem leading to advances in image and/or signal reconstruction. It is easy to show that a subsequence of a moment sequence may not be a moment sequence. Conditions are obtained to show how rigid the space of sub-moment sequences is and necessary and sufficient conditions for a sequence to be a sub-moment sequence is established. A deep connection between the sub-moment measures and the moment measures is derived and the determinacy of the moment and sub-moment problems are related. This problem is further related to completion of positive Hankel matrices.

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### Asymptotics of orbits of a kolmogorov type planar vector field with a fixed newton polygon

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Using the Newton polygon technique we show that the orbits of a Kolmogorov type planar vector field, consisting of a finite sum of power terms, have power asymptotics while tending to the equilibria on the boundary of the Poincare sphere.

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### Systems of polynomial ODE's as a tool for improving the efficiency of numerical methods

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We show that the efficiency of computational methods can be greatly increased by employing a novel substitution approach to any arbitrary ODE. Parker and Sochacki showed that any ODE can be made into a system of polynomial ODE's, examples will be given and the speed and accuracy of methods such as Runge-Kutta will be tested using this approach. These speeds and accuracies are then compared with the same numerical methods, using the original ODE, as well as with Parker and Sochacki's Modified Picard Method (PSM). The results show that all tested methods are improved by polynomial projection of the ODE's and that PSM is even faster and more accurate for a very large range of problems. A GUI implementing PSM will be demonstrated.

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### Fast generation algorithms for the Adomian polynomials

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We present different recurrence algorithms for the multivariable Adomian polynomials. These algorithms are comprised of simple recurrence formulas and they are straightforward to implement

in any symbolic software. Recurrence algorithms for the one-variable Adomian polynomials are derived as special cases. In particular, the recurrence process of the reduced polynomials $C(n,k)$, where $C(n,k)$ comprise of the Adomian polynomials $A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C(n,k)$, does not involve the differentiation operation, but significantly only the arithmetic operations of multiplication and addition are involved. The MATHEMATICA program generating the Adomian polynomials based on the new, fast algorithms is designed. We also demonstrate that the Adomian polynomials can be applied to solve the nonlinear fractional differential equations.

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**The unifying view on ordinary differential equations and automatic differentiation, yet with a gap to fill**

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Among various instruments of approximation representing solutions of Ordinary Differential Equations (ODEs), the Taylor expansions play a unique role. This is because an explicit system of ODEs by itself is a tool for computing n-order derivatives of the solution and therefore delivering the Taylor expansions of the solution at multiple points of the phase space.

This capability by ODEs to generate Taylor expansions relies however on availability of the rules of n-order differentiation and their computational efficiency. Availability and efficiency of the rules of n-order differentiation take place only for a particular sub-class of holomorphic functions. Those are the so called generalized elementary functions, i.e. functions representable as solutions of rational ODEs.

Continuation of (generalized) elementary functions via integration of its ODEs not necessarily expands them into each and every point where these functions exist and are holomorphic. Some entire functions are suspects for being elementary everywhere except isolated unreachable points - the points of their "removable" or "regular" singularity. All the above mentioned and a few other issues are interdependent and complementary to each other. The merger of them into one theory is called the Unifying view on ODEs and AD.

However there is a gap in this otherwise coherent view: an open statement, the Conjecture about the possibility to convert a rational system of ODEs at a regular point into one n-order rational ODE regular at the same point. The Conjecture is important because the question of equivalency of two competing definitions of elementary functions (and a few other open statements) depend on the Conjecture.

The report therefore presents the setting and a few known facts concerning the Conjecture as an invitation for everybody to resolve it and fill the gap in this theory.

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**Different differential equations with the same solution**

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**J. Sochacki**

We show how different differential equations with the same solution can have different errors when solved using the identical numerical method. This is unexpected, since virtually every discussion of numerical method errors gives them in terms of the solution. We show this behavior with a number of simple examples, including the classic pendulum, where energy should be conserved. Rewriting differential equations in polynomial form is shown to usually be the best option, and we conclude with a discussion of symplectic solvers compared to the Power Series Method.

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**Efficient recurrence relations for univariate and multivariate Taylor series coefficients**

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The efficient use of Taylor Series depends, not on symbolic differentiation, but on a standard set of recurrence formulas for each of the elementary functions and operations. These relationships are often rediscovered and restated, usually in a piecemeal fashion. We seek to provide a fairly thorough and unified exposition of efficient recurrence relations in both univariate and multivariate settings. Explicit formulas all stem from the fact that multiplication of functions corresponds to a Cauchy product of series coefficients, which is more efficient than the Leibniz rule for nth-order derivatives. This principle is applied to function relationships of the form h'=v*u', where the prime indicates a derivative or partial derivative. Each elementary transcendental function corresponds to an equation, or pair of equations, of that form h'=v*u'. A geometric description of the multivariate operation helps clarify and streamline the computation for each desired multi-indexed coefficient. Several different perspectives on these relationships will be mentioned from a Chinese algorithm in year 1247 to the Differential Transform Method now active in Asian literature.

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**A generalization of Darboux's method**

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There is a classical method started by Darboux in 1878 about the computation of a special class of first integrals(called Darboux first integrals) of polynomial differential systems. These Darboux first integrals are the products of powers of polynomials (often called Darboux polynomials). Note that the zero set of the Darboux polynomials define invariant algebraic curves of the corresponding vector field. So, algebraic curves and its multiplicity are the key points in order to construct the Darboux first integrals. More recently this method has been generalized by several authors like Jouanolou, Singer, Schlomiuk, Llibre, Christopher, Zhang among others and is related with some aplications concerning limit cicles and bifurcation problems. Several inverse problems in dimension two have also been considered. In this talk we present necessary and sufficient conditions for the existence of Darboux first integrals and we also present a generalization for the nonautonomous case.

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**Solving ODEs using PSM and trees**

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Using the polynomial projections and power series method (PSM) developed by Parker and Sochacki and others, an implementation has been developed that computes numerical solutions to ODEs to any degree of accuracy using PSM. ODEs are parsed into tree structures, and the trees are traversed with calls to a small library of functions derived for this purpose. The code that solves ODEs using this method will be presented and a discussion of how the form of the algorithm lends to the optimization of PSM will be given.

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**Padé approximants and pole extraction near singular points.**

**Joseph Rudmin**
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Ed Parker and James Sochacki have rediscovered a powerful way of finding polynomial approximations to systems of differential equations. This Parker Sochacki method always finds the Taylor Series approximation to a given order, if such approximation exists, and gives absolute error limits. For most practical applications, a Padé approximant derived from the Taylor Series provides better fit than the Taylor Series. However, both Taylor Series and Padé Approximants have difficulty modeling poles in the solution. Often one can best model a pole by a change of variable, where the variable explicitly contains the pole. The change of variable can be found from the differential equations by eliminating the highest order feedback loop in the Parker Sochacki approximation, thus simplifying those equations.

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**Mathematical modeling problems that are polynomial ODEs**

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There are many force problems that can be posed as systems of polynomial ODEs with polynomial energy conditions. Many numerical methods, such as Newton's Method, the Method of Steepest Descent, Inverses of Functions and Pade Approximants can be expressed as systems of polynomial ODEs. Several of these will be presented and it will be discussed how Picard Iteration and/or power series can be used to solve these problems to within a priori error bounds.

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