Special Session 6: Direct and Inverse Problems in Phase Field Systems and Related Subjects

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In the last decades, a number of models of phase field type appeared in the literature. They are often well motivated by physical considerations (e.g., thermodynamics, continuum mechanics, etc.). For that reason, boundary value problems for the corresponding PDE systems have been deeply studied and many mathematical results are known that deal with several viewpoints, e.g., well-posedness, asymptotic analyses, identification of unknown functional parameters, etc. This session aims presenting results on such a subject. On the other hand, some studies on completely different problems, e.g., on abstract equations in Banach spaces, can have applications in the above topics. Therefore, talks on a wider field are welcome.

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Well-posedness results for a model of contact with adhesion

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We deal with a contact problem with adhesion between a viscoelastic body and a rigid support. We refer to a model recently proposed by M. Frémond combining the damage theory in continuum mechanics with the theory of the unilateral contact. The resulting PDE's system is given by a balance equation for macroscopic movements coupled with an equilibrium equation describing the evolution of the adhesion on the contact surface. In a joint work with E. Bonetti and R. Rossi, we investigate the well-posedness of the related initial and boundary value problem, the continuous dependence on the data, and the long-time behaviour of the solutions.

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On a model for phase transitions with entropy equation and thermal memory conductivity

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This talk deals with a thermodynamic model describing phase transitions with thermal memory in terms of an entropy equation and a momentum balance for the microforces. This phase field model leads to a boundary value problem for a system of partial differential equations that will be presented and discussed. From the analytical point of view, in a recent joint work with E. Bonetti, M. Fabrizio and G. Gilardi we investigate existence and uniqueness of the solutions, continuous dependence on the data, regularity, and long-time behaviour. Actually, we can prove existence and uniqueness of a global solution to the Cauchy problem for the system of evolution equations. We also consider the omega-limit set of the trajectories and characterize the omega-limit points as solutions of a suitable stationary problem.

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A Transmission Problem in a Thin Layer

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We study a transmission problem in a thin layer by writing it as an abstract differential equation.

In an UMD Banach space X we consider the problem

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x) & x \in (-1,0) \\ u_+''(x) + Au_+(x) = -g_+(x) & x \in (0,\delta) \\ u_-(-1) = 0,\ u_+'(\delta) = 0 \\ u_-(0) = u_+(0),\ u_-'(0) = \delta^{-1}u_+'(0) \end{cases}$$

where A is a linear closed densely defined operator in X, such that $-A$ is positive and has a bounded H^∞ functional calculus. These hypotheses are satisfied if $X = L^p$ $(1 < p < \infty)$ and A is the realization in X of an elliptic operator with suitable boundary conditions.

We prove existence and uniqueness of the solution in the sense of L^p of this problem and study the behaviour of the solution when the thickness δ of the layer tends to 0.

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An identification problem for a degenerate differential equation of the second order

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We are concerned with a degenerate second order identification problem in a Banach space of the type

$$\begin{aligned}
&\frac{d}{dt}(M\frac{dy}{dt}) + L\frac{dy}{dt} + Ky = f(t)z,\ 0 \le t \le \tau,\\
&y(0) = y_0,\ (My')(0) = My_1,\\
&\Phi[My(t)] = g(t),\ 0 \le t \le \tau,
\end{aligned}$$

where (y, f) is the unknown and the linear closed operator M may admit no bounded inverse. Suitable hypotheses on the involved operators are made in order to reduce the given problem to a first order system.

Some applications to partial differential equations are indicated.

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Nonlinear degenerate parabolic equations for a thermohydraulics model

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In this talk, we shall study an initial-boundary value problem for a system of second order partial differential equations. This system is consisted by the Navier-Stokes equations and nonlinear heat equation. More precisely we impose a nonlinear heat flux associated with an arbitrary maximal monotone graph with Neumann boundary condition. We establish a necessary and sufficient conditions on given data for the existence of the solution.

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Phase field models for multicomponent alloy solidification

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We present a phase field model for phase transformations in multicomponent alloys. First we discuss how it is possible to approximate sharp interface models to second order in the interfacial thickness. Then we introduce novel potential energies for multi-phase systems which allow it to more accurately model surface energy in multiphase systems. Finally we discuss how the system can be coupled to the Navier-Stokes equations.

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Convergence of a singular phase field system with memory to phase relaxation

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A singular phase system with memory for the absolute temperature ϑ and the order parameter χ has been recently considered. Namely, the equation for temperature is $\partial_t(\vartheta - \lambda(\chi)) - \Delta(\vartheta + k * \vartheta) = R$, while the dynamics for χ is ruled by $\partial_t\chi - \varepsilon\Delta\chi + W'(\chi) = -\vartheta\lambda'(\chi)$. In the above equations, λ is a real function on the real line, k is a memory kernel, R is a given source term, ε is a positive parameter, and W' is the derivative of a double well potential. More generally, the convex part of W might be infinite somewhere and assumed to be just lower semicontinuous. In such a case, its derivative is replaced by its subdifferential. Finally, the system is complemented with initial and suitable boundary conditions. In a joint work with E. Rocca (Milan), the asymptotic analysis as ε tend to 0 is performed. One clearly expects that the limit problem is the phase relaxation system obtained simply taking $\varepsilon = 0$ in the above system and forgetting the boundary conditions for χ. Well, this is essentially true and it is not difficult to prove it in the simplest cases. However, a correct proof is not known for rather general λ and W. Serious technical troubles arise in identifying the limits of the nonlinear terms, indeed. The aim of the talk is giving an outline of our results.

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The Heat Equation with Dynamic Linear and Nonlinear Boundary Conditions

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Of concern is the heat equation

$$u_t = \bigtriangledown \cdot (a(x) \bigtriangledown u)$$

on a bounded domain Ω in $\mathbb{R}^n$, with $\partial\Omega$ of class C^2 with dynamic boundary conditions of the form

$$u_t + b(x)\frac{\partial u}{\partial n_a} + c(x)u = 0$$

or with dynamic nonlinear boundary conditions of the form

$$u_t + b(x)\frac{\partial u}{\partial n_a} + c(x)u \in \beta(x,u).$$

Here $a \in C^2\left(\overline{\Omega}\right)$, $a > 0$ on $\overline{\Omega}$, $b, c \in C(\partial\Omega)$, $b > 0$, $\beta(\cdot,r)$ is a maximal monotone graph satisfying $\beta(\cdot,0) \ni 0$, and $\frac{\partial}{\partial n_a}$ is the corresponding conormal derivative. We derive these boundary conditions from basic physical principles and give their physical interpretation. We show that the above problems are well-posed on $X_p = L^p(\Omega, dx) \oplus L^p(\partial\Omega, w(x)\,dS)$ for a suitable weight function w, $1 \le p < \infty$ and on $X_\infty = C\left(\overline{\Omega}\right)$. Moreover, we prove estimates to show that the heat equation with nonlinear dynamic boundary conditions has the optimal regularity properties on these spaces.

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Critical constants and nonexistence

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Consider the parabolic equation

$$\partial u/\partial t = \Delta u + V(x)u \qquad (HE)$$

where $V(x) = c/|x|^2$ and $x \in \mathbb{R}^N$. Then (HE) has positive solutions if and only if $c \le ((N-2)/2)^2$. Recently this (1984) result has been extended in many ways. The Laplacian can be replaced by a second order uniformly elliptic operator with non-smooth coefficients or certain degenerate operators, such as the Laplacian on certain Carnot groups (including the Heisenberg group), or weighted nonlinear operators related to the p-Laplacian or the fast diffusion operator, or Baouendi-Grushin operators, et al. In each case one must find the appropriate (linear or nonlinear) operator V, and a critical constant emerges. Ingredients in the proofs include energy methods, and Hardy and Harnack inequalities.

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Convergence to a stationary state for solutions of semilinear parabolic inverse problems

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We consider an inverse problem, with an abstract semilinear parabolic integrodifferential equation. The convolution kernel, together with the solution of the parabolic problem, is unknown. A supplementary condition, in the form of a certain functional applied to the (unknown) solution is given. The aim is to reconstruct the solution, together with the convolution kernel. This problem has been recently considered by many authors from many points of view. Our aim is to look for sufficient conditions, ensuring that the solution is globally defined and converges to a stationary state, as time t goes to $+\infty$. The main tools are maximal regularity results for abstract parabolic systems in an L^1-setting, together with some standard facts concerning boundedness of solutions of global linear parabolic abstract systems.

The results were obtained in collaboration with F. Colombo.

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A class of doubly nonlinear systems for phase transitions

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A class of doubly nonlinear evolution systems is considered. The typical one, which we discuss in this talk, is of the form

$$\partial\varphi_u(u') + \partial\psi(u) + G(u) \ni f(t)$$

in a real Hilbert space H, where $\varphi_u(\cdot)$ is a proper l.s.c. convex function on H with parameter $u \in H$ and $psi(\cdot)$ is a proper l.s.c. convex function on H having compact level sets. Also, $G(\cdot)$ is a nonlinear (unbounded) operator in H which satisfies a Lipschitz continuity property ($\psi-$Lipschitz continuity) specified by the convex function ψ. We shall give an abstract existence result for the Cauchy problem, and apply it to irreversible phase transition problems with the unknown-dependent irreversibility, for instance,

$$u_t + w_t - \kappa\Delta u = h(w,u),$$

$$\partial I_0(w_t - p(w,u)) - \tau\Delta w \ni f(w,u),$$

where $I_0(\cdot)$ is the indicator function on $\mathbf{R}^+$.

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Existence and asymptotic analysis of a phase field model for supercooling

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In a joint work with Olaf Klein and Riccarda Rossi an existence result for an initial-boundary value problem is proved. The evolution system under consideration can model a perturbation of a phase transition phenomenon with supercooling effects. The proof consists of the following steps. First, study an abstract doubly nonlinear evolution equation by means of a time discretization procedure. Next, introduce a suitable approximation. Then obtain the local (in time) existence of the regularized problem through a fixed point argument. Finally, pass to the limit with respect to the regularization parameter, thanks to some suitable *a priori* estimates. An asymptotic analysis is also performed, when the perturbation coefficient ε goes to 0, leading to an existence result, in the framework of Young measures, for a slight modification of the original unperturbed problem.

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On a degenerate problem in porous media

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The purpose of the paper is to study Richards' equation

$$C(h)\frac{\partial h}{\partial t} - \nabla\cdot(k(h)\nabla h) + \frac{\partial k(h)}{\partial x_3} = f \text{in } \Omega\times(0,T),$$

with initial and boundary data. This equation which degenerates for $h \geq 0$, when $C(h)$ vanishes, describes in particular an infiltration process in an unsaturated porous medium. By suitable function transformations we obtain an equivalent diffusive form

$$\frac{\partial \theta}{\partial t} - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{in } \Omega\times(0,T),$$

in which a multivalued function β^* is introduced to model the evolving free boundary process, that is the simultaneous unsaturated and saturated flow. Various hypotheses upon the analytical properties of Richards' equation coefficients influence the behaviour of β^* around the saturation point, and imprints a more linear or a more nonlinear character to the model. We are concerned with the proof of the existence, uniqueness and properties of the solution, especially for the strongly nonlinear case, in the framework of the theory of evolution equations with m-accretive operators in Hilbert spaces.

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Generators of Feller semigroups with coefficients depending on parameters

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Jerome A. Goldstein and Rosa Maria Mininni

We present some of the results obtained in a joint work with Jerome A. Goldstein and Rosa Maria Mininni, where different realizations of the operators $L_{\theta,a}u(x) = x^{2a}u''(x) + (ax^{2a-1} + \theta x^a)u'(x)$, $\theta \in \mathbf{R}$, $a \in \mathbf{R}$, on suitable spaces of bounded continuous real valued functions on $\mathbf{R}$, or on $\mathbf{R}_+$, are considered. Our main aim is to show that, for suitable values of a, the closures of these operators are perturbations of squares of generators of groups of operators. Connections with Romanov's formula will be also shown.

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Solvability for phase field systems of Penrose-Fife type associated with nonlinear diffusions

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Let $\Omega \subset \mathbf{R}^N$ $(1 \leq N \leq 3)$ be a bounded domain with a smooth boundary. Here, for any fixed $p \in [1,2)$, let us consider the following phase field system of Penrose-Fife type:

$$\begin{cases} \partial_t(\theta + \lambda(w)) - \triangle\left(-\dfrac{1}{\theta}\right) \\ \qquad = f(x,t), \quad (x,t) \in \Omega\times(0,+\infty), \\ \partial_t w(t) + \kappa\partial\varphi^p(w(t)) + \beta(w(t)) + g(w(t)) \\ \qquad \ni -\dfrac{\lambda'(w(t))}{\theta(t)} \text{ in } L^2(\Omega), \ t > 0; \end{cases} \qquad (*)$$

subject to suitable initial-boundary conditions. In the context, θ is the absolutely temperature, and w is the nonconserved order parameter. λ, g are given smooth functions, and λ' is the derivative of λ. β is a maximal monotone in $\mathbf{R}\times\mathbf{R}$ with the domain $D(\beta)$. $\partial\varphi^p$ is the subdifferential of a proper l.s.c. convex function φ^p on $L^2(\Omega)$, defined as:

$$\varphi^p(z) := \begin{cases} \displaystyle\int_\Omega |\nabla z|^p \, dx, \text{ if } p > 1, \\ \displaystyle\int_\Omega |Dz| \text{ (total variation), if } p = 1, \end{cases}.$$

$z \in L^2(\Omega)$, f is a given forcing term, and κ is a given positive constant. In this talk, we focus on the following two cases of β:

- $\overline{D(\beta)}$ is compact,
- $\beta \equiv 0$ in $\mathbf{R}$;

to discuss about the solvability of $(*)$ in these cases. Consequently, in each case, the existence theorem of (weak) solutions and some related corollaries will be reported, under appropriate assumptions for λ, g and β.

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